

ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR RANDOM NON-UNIFORMLY EXPANDING MAPS

VITOR ARAUJO AND JAVIER SOLANO

ABSTRACT. We prove existence of (at most denumerable many) absolutely continuous invariant probability measures for random one-dimensional dynamical systems with asymptotic expansion. If the rate of expansion (Lyapunov exponents) is bounded away from zero, we obtain finitely many ergodic absolutely continuous invariant probability measures, describing the asymptotics of almost every point. We also prove a similar result for higher-dimensional random non-uniformly expanding dynamical systems. In both cases our method deals with either critical or singular points for the random maps.

1. INTRODUCTION

In this work we study the existence of absolutely continuous invariant probability measures for the random iteration of maps of the interval, or of a compact manifold, which have positive Lyapunov exponents but can also have critical points or singularities. We also obtain a decomposition of each absolutely continuous invariant measure into at most denumerably many absolutely continuous ergodic components. This can be seen as an extension of the results of Pelikan [25], Morita [24] and Buzzi [9] which deal with random iterations of piecewise expanding maps.

It is well-known that the dynamics of random maps can be modeled by a skew-product map where the “noise” is driven by the ergodic base transformation. This is the general form of a Random Dynamical System; see [7, Definition 1.1.1]. Hence our results can also be seen as a study of the dynamics of skew-product whose maps along the one-dimensional fibers have critical points or discontinuities, positive Lyapunov exponents and very weak conditions on the base transformation. We mention the work of Denker and Gordin [13] together with Heinemann [14] where equilibrium states for random bundle dynamics were studied under the assumption of expansion along the fibers.

As an example of application of our results we present the following. Let us consider the map $\varphi(\theta, x) = (\alpha(\theta), f(\theta, x))$ with $\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ a continuous map with an ergodic α -invariant probability measure ν ; and $f_\theta(x) = a(\theta) - x^2$ for $a(\theta)$ continuous so that φ is well-defined, and \mathfrak{m} the Lebesgue measure on the interval $[-2, 2]$. We use the notation $\varphi^n(\theta, x) = (\alpha^n(\theta), f_\theta^n(x))$.

Corollary 1.1. *Assume that there exists $\lambda > 0$ such that, for $(\nu \times \mathfrak{m})$ -almost every (θ, x) ,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df_\theta^n(x)| \geq \lambda$$

Then φ admits finitely many ergodic invariant probability measures absolutely continuous with respect to $\nu \times \mathfrak{m}$. Moreover, $(\nu \times \mathfrak{m})$ -almost every (θ, x) belongs to the basin of one of these measures.

The weak assumptions of the dynamics of the base map allows us to state our results in the setting of random dynamical systems; see Corollary 1.3 in Subsection 1.1.2 for details.

This work can also be seen as a generalization of the earlier work of Keller [20] which proves that for maps of the interval with finitely many critical points and non-positive Schwarzian derivative, existence

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of absolutely continuous invariant probability is guaranteed by positive Lyapunov exponents, i.e.,

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log |Df^n(x)| > 0 \quad \text{on a positive measure set of points } x.$$

On the one hand, related result in were obtained by Alves, Bonatti and Viana. They show that every non-uniformly expanding local diffeomorphism away from a non-degenerate critical/singular set, on any compact manifold, admits a finite number of ergodic absolutely continuous invariant measures describing the asymptotics of almost every point. The notion of non-uniform expansion means that

$$(1.1) \quad \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < 0 \quad \text{almost everywhere.}$$

Some control of recurrence to this critical/singular set must be assumed to construct the absolutely continuous invariant measures. This assumption is usually rather difficult to verify.

The main known example of maps satisfying the conditions of the result of Alves, Bonatti and Viana are the *Viana maps*. These maps were introduced by Viana [32] and studied by many authors, e.g. [3, 4, 6, 11, 28] among others. The maps are skew-products $\varphi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \times \mathbb{Y}$, $(\theta, x) \mapsto (\alpha(\theta), f(\theta, x))$, with α being a uniformly expanding circle map and the maps on the fibers being quadratic maps of the interval. The central direction along \mathbb{Y} is dominated by the strong expansion of the base dynamics along \mathbb{X} . For an open class of these maps, Viana [32] proved the positiveness of the Lyapunov exponents and Alves [3] proved the existence of an absolutely continuous invariant measure.

Extensions of the above mentioned results were obtained, among others, by Pinheiro [26], and by one of the authors [30] but, in all cases, either non-uniform expansion (1.1) in all directions, or a weaker form of hyperbolicity (partial hyperbolicity) is demanded. The critical/singular set is also assumed to be non-degenerate. In a remarkable work, Tsujii [31] proves results in this line for generic partially hyperbolic endomorphisms on compact surfaces.

On the other hand, for piecewise expanding maps in higher dimensions, the existence of absolutely continuous invariant measures was obtained by Adl-Zarabi [1], Buzzi [10], Gora-Boyarsky [15], Keller [19] and, among other, Saussol [27]. Again the authors assume uniform expansion with strong expansion rates together with certain boundary conditions on the pieces of the domain where the transformation is not expanding.

Our results demand no partial hyperbolicity or domination conditions and we put no restriction on the dynamics of the base of the skew-product, other than almost everywhere continuity and the existence of an invariant ergodic probability measure. We do not require non-uniform expansion (1.1) in all directions, nor the non-degenerate conditions of the critical set. This allows to state our results for random dynamical maps. Along multidimensional fibers (i.e. the dimension of the space \mathbb{Y}), we do demand non-uniform expansion and a control of the recurrence to the singular/critical set. Along one-dimensional fibers (i.e., the case where \mathbb{Y} is the interval) with f_θ having negative Schwarzian, we assume non-uniform expansion only: we do not assume slow recurrence. In particular, the base dynamics can have no absolutely continuous invariant measure with respect to some natural volume form, as we present in some examples. Under these mild conditions we prove the existence of at most denumerable many invariant probability measures absolutely continuous along the fibers.

1.1. Statements of results. For a topological space X we denote by \mathcal{B}_X the Borel σ -algebra on X . The main setting is the following: let \mathbb{X} and \mathbb{Y} be a separable metrizable and complete (i.e., Polish) topological spaces. Let us consider the skew-product map

$$\begin{aligned} \varphi : \mathbb{X} \times \mathbb{Y} &\longrightarrow \mathbb{X} \times \mathbb{Y} \\ (\theta, x) &\longmapsto (\alpha(\theta), f(\theta, x)). \end{aligned}$$

We assume that φ is at least measurable with respect to the Borel σ -algebra $\mathcal{B}_\mathbb{X} \times \mathcal{B}_\mathbb{Y}$ (which equals $\mathcal{B}_{\mathbb{X} \times \mathbb{Y}}$ since both \mathbb{X} and \mathbb{Y} are separable metric spaces; see e.g. [8, Appendix M.10]).

1.1.1. *One dimensional fibers.* We consider $\mathbb{Y} = I_0$ a compact interval. For $\theta \in \mathbb{X}$, $f_\theta : I_0 \rightarrow I_0$, $x \mapsto f(\theta, x)$ is an interval map, possibly with critical points and discontinuities. We denote by \mathcal{C}_θ and \mathcal{D}_θ the set of critical points and discontinuities, respectively, of f_θ , for every $\theta \in \mathbb{X}$. We also use the notations $\mathcal{C} = \{(\theta, x) \in \mathbb{X} \times I_0; x \in \mathcal{C}_\theta\}$ and $\mathcal{D} = \{(\theta, x) \in \mathbb{X} \times I_0; x \in \mathcal{D}_\theta\}$.

We assume throughout that the discontinuities \mathcal{D}_θ of the interval map f_θ are in the interior of I_0 , and that the lateral limits exist at each $x \in \mathcal{D}_\theta$; see condition (H_4^*) in what follows.

We assume also that

(H_1) $p := \sup\{\#(\mathcal{C}_\theta \cup \mathcal{D}_\theta), \theta \in \mathbb{X}\} < \infty$ and $\Gamma := \sup\{\partial_x f(\theta, x), (\theta, x) \notin \mathcal{D}_\theta\} < \infty$. The set

$$\mathcal{S} = \{(\theta, x) \in \mathbb{X} \times I_0; x \in \mathcal{C}_\theta \cup \mathcal{D}_\theta\}$$

is measurable (i.e. it belongs to $\mathcal{B}_\mathbb{X} \times \mathcal{B}_{I_0}$).

(H_2) $\alpha : \mathbb{X} \rightarrow \mathbb{X}$ is a measurable map with an ergodic invariant probability measure ν such that $\nu(\mathcal{D}_\alpha) = 0$.

The assumption on the discontinuity set is a natural condition to study the φ -invariance of weak* accumulation points of dynamically defined probability measures. Let us consider the map

$$F : \mathbb{X} \rightarrow B(I_0) \quad \theta \mapsto f_\theta : I_0 \rightarrow I_0$$

where $B(I_0)$ is the family of measurable maps from I_0 to I_0 with the uniform norm:

$$\|F(\tilde{\theta}) - F(\theta)\| = \sup_{x \in I_0} |f_{\tilde{\theta}}(x) - f_\theta(x)|.$$

We write \mathcal{D}_F for the set of discontinuities of the map F . We further assume some regularity of the map F

(H_3) $\nu(\mathcal{D}_F) = 0$.

We deal with two situations:

(H_4) the maps f_θ are C^3 , $Sf_\theta \leq 0$, for every $\theta \in \mathbb{X}$ (here Sf_θ is the Schwarzian derivative of f_θ) and the derivatives of $\{f_\theta\}_{\theta \in \mathbb{X}}$ are equicontinuous.¹

(H_4^*) we have $\mathcal{D}_\theta \neq \emptyset$ for some $\theta \in \mathbb{X}$. Writing $\mathcal{D}_\theta = \{q_1(\theta) \leq \dots \leq q_{d(\theta)}(\theta)\}$ (this may be the empty set for some values of $\theta \in \mathbb{X}$) for every $\theta \in \mathbb{X}$, we assume that f_θ is C^3 diffeomorphism and $Sf_\theta \leq 0$ restricted to $(q_i(\theta), q_{i+1}(\theta))$ for all $i = 0, 1, \dots, d(\theta)$, where we set $q_0 = \inf I_0$ and $q_{d(\theta)+1} = \sup I_0$ to be the endpoints of I_0 .

Writing $\mathcal{D} = \{(\theta, x) : x \in \mathcal{D}_\theta, \theta \in \mathbb{X}\}$ we also assume that for every $\ell \in \mathbb{Z}^+$ there exists a neighborhood V of $\overline{\mathcal{D}}$ such that

$$\varphi^k(V) \cap V = \emptyset \quad \text{for every } k = 1, \dots, \ell.$$

We write, here and in the rest of the paper, \overline{C} for the topological closure of a subset $C \subset \mathbb{X} \times I_0$.

This setting models similar maps as in [16, 30], but without expansion assumptions on the base, and we also admit discontinuities but with strong non-recurrence assumptions. This non-recurrence property can be deduced, as in Example 2, if every sequence z_k in $\mathbb{X} \times I_0$ tending to \mathcal{D} is sent to a sequence $\varphi(z_k)$ tending to a forward invariant subset disjoint from $\overline{\mathcal{D}}$; a sort of Misiurewicz condition, but this time on the images of a discontinuity set.

We say that φ has *positive Lyapunov exponents along the vertical direction* according to $\nu \times m$, if

$$(1.2) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df_\theta^n(x)| > 0 \quad (\nu \times m) - \text{almost all } (\theta, x)$$

where m denotes the Lebesgue measure on I_0 and we use the convention

$$f_\theta^k(x) := f_{\alpha^{k-1}(\theta)} \circ \dots \circ f_{\alpha^1(\theta)} \circ f_\theta(x)$$

for every $\theta \in \mathbb{X}$, $x \in I_0$. We say that φ has *positive Lyapunov exponents along the vertical direction* according to $\nu \times m$, on the subset Z , if (1.2) holds for $\nu \times m$ -a.e. $(\theta, x) \in Z$.

¹The equicontinuity can be replaced by the following condition: given $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - \mathcal{C}_\theta| < \delta$ then $|f'_\theta(x)| < \epsilon$, for all $\theta \in \mathbb{X}$. This is used in the proof of Theorem 4.1.

We recall that for an ergodic φ -invariant probability measure, its *ergodic basin* is the set

$$B(\mu) = \left\{ \omega = (\theta, x) \in \mathbb{X} \times \mathbb{Y} : \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(\varphi^j(\omega)) = \int g d\mu \text{ for each } g \in C^0(\mathbb{X} \times \mathbb{Y}, \mathbb{R}) \right\}.$$

Our main result in this setting is the following

Theorem A. *Let $\varphi : \mathbb{X} \times I_0 \rightarrow \mathbb{X} \times I_0$ be a skew-product as above satisfying (H_1) , (H_2) , (H_3) and (H_4) (or (H_4^*)). Assume that φ has positive Lyapunov exponents along the vertical direction on the subset Z .*

Then φ admits an at most denumerable family $\{\mu_i\}_{i \in \mathbb{I}}$ of ergodic invariant probability measures absolutely continuous with respect to $\nu \times m$. Moreover $\nu \times m$ -almost every $(\theta, x) \in Z$ belongs to the basin of some $\mu_i, i \geq 1$.

Note that the existence of an invariant measure for the base dynamics (see condition (H_2)) is not a restriction in the theorem. Indeed, any φ -invariant measure absolutely continuous (with respect to $\mu_{\mathbb{X}} \times m$, where $\mu_{\mathbb{X}}$ is a measure on $\mathcal{B}_{\mathbb{X}}$) induces an α -invariant measure absolutely continuous (with respect to $\mu_{\mathbb{X}}$).

In the case that the rate of expansion is bounded away from zero, we have a stronger result.

Corollary B. *Let $\varphi : \mathbb{X} \times I_0 \rightarrow \mathbb{X} \times I_0$ be a skew-product as above satisfying (H_1) , (H_2) , (H_3) and (H_4) (or (H_4^*)). Assume that there exists $\lambda > 0$ such that the limit in (1.2) is greater than 2λ , for a.e. $(\theta, x) \in Z$. Then φ admits finitely many ergodic invariant probability measures absolutely continuous with respect to $\nu \times m$, whose basins cover Z , up to a $\nu \times m$ -zero measure set.*

1.1.2. Random dynamical systems interpretation. Let $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \nu)$ be a probability space and let α be an ν -preserving measurable map on \mathbb{X} . A random dynamical system f on the measurable space $(\mathbb{Y}, \mathcal{B}_{\mathbb{Y}})$ over $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \nu, \alpha)$ is generated by mappings $f_{\theta}, \theta \in \mathbb{X}$, so that the map $(\theta, x) \rightarrow f_{\theta}(x)$ is measurable and it holds the cocycle property $f_{\theta}^{n+m} = f_{\alpha^m(\theta)}^n \circ f_{\theta}^m$ (see [7, Definition 1.1.1]). The associated random orbits are x_0, x_1, \dots , where $x_0 \in \mathbb{Y}$ and $x_{n+1} = f_{\alpha^n(\theta)}(x_n)$. This random dynamical system (RDS for short) is denoted by $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \nu, \alpha, f)$.

In general there is no common measure invariant for all the maps $f_{\theta}, \theta \in \mathbb{X}$. But one can ask whether there exists a measure (or a finite number of measures) describing the asymptotics of almost all random orbits, in the sense defined to follow. Let us denote by δ_x the Dirac measure at x .

Definition 1.2. A probability measure μ on \mathbb{Y} is SRB for the RDS $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \nu, \alpha, f)$ if, for ν -almost every $\theta \in \mathbb{X}$, the set $RB_{\theta}(\mu)$ of points $x \in \mathbb{Y}$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_{\alpha^{k-1}(\theta)} \circ \dots \circ f_{\theta}(x)} \longrightarrow \mu$$

has positive Lebesgue measure. We call $RB_{\theta}(\mu)$ the random basin of μ .

One can associate to the random map f the skew product $\varphi : \mathbb{X} \times \mathbb{Y} \cup, (\theta, x) \mapsto (\alpha(\theta), f_{\theta}(x))$. Note that, a φ -invariant measure μ with marginal ν , that is, such that $\mu(A \times I_0) = \nu(A)$ for every ν -measurable $A \subset \mathbb{X}$, is an *invariant measure for the random dynamical system* $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \nu, \alpha, f)$; see [7, Definition 1.4.1]. All the φ -invariant measures obtained in Theorems A and D are of this type; see Lemma 3.1 in Section 3.

We say that the random map $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \nu, \alpha, f)$ is a

- *random non-uniformly expanding map on I_0* if \mathbb{X} is a Polish space, $\mathbb{Y} = I_0$ and there exists $\lambda > 0$ such that the limit in (1.2) is greater than 2λ , for $(\nu \times m)$ -almost all (θ, x) .
- *admissible random non-uniformly expanding map on I_0* if it is a random non-uniformly expanding map on I_0 and the associated skew-product satisfies (H_1) , (H_2) , (H_3) and (H_4) (or (H_4^*)).

We can state similar results to Theorem A and Corollary B in the setting of random non-uniformly expanding maps, since that the associated skew-product satisfies the conditions of these results. Moreover, inspired by one result of Buzzi [9, Theorem 0.5], we can state the following probabilistic consequence of our results.

Theorem C. *Any admissible random non-uniformly expanding map on I_0 admits a finite number of SRB measures. Moreover, the SRB measures are absolutely continuous and, ν -almost surely, the union of their random basins has total Lebesgue measure.*

We observe that if $\mathbb{X} = \Sigma^{\mathbb{N}}$, where Σ is an at most countable set, then \mathbb{X} is totally disconnected. In addition, setting $f_\theta = f_{\pi(\theta)}$ where $\pi : \mathbb{X} \rightarrow \Sigma^k$ is a projection on the first k -symbols of $\theta \in \mathbb{X}$, and α the left shift of $\Sigma^{\mathbb{N}}$ we have both $\mathcal{D}_\alpha = \emptyset$ and $\mathcal{D}_F = \emptyset$, since f_θ depends only on finitely many coordinates of the point $\theta \in \mathbb{X}$ (the map $F : \mathbb{X} \rightarrow B(I_0)$ is locally constant).

Hence we obtain the following as a immediate corollary of Theorem C.

Corollary 1.3. *Let $f_i : I_0 \rightarrow I_0, i \in \Sigma$ be a countable family of maps of the quadratic family, $\mathbb{X} = \Sigma^{\mathbb{N}}$ and $\alpha : \mathbb{X} \rightarrow \mathbb{X}$ be the left shift with some ergodic α -invariant probability measure ν .*

If $(\Sigma^{\mathbb{N}}, \mathcal{B}_{\Sigma^{\mathbb{N}}}, \nu, \alpha, f)$ is a random non-uniformly expanding map on I_0 , then it admits a finite number of SRB measures. The SRB measures are absolutely continuous and the union of their random basins has total Lebesgue measure ν -a.e.

Similar results holds for families of maps satisfying the non-uniformly expanding conditions of the following Section 1.1.3 with higher-dimensional fibers.

1.1.3. Higher-dimensional fibers. Assuming a condition of slow recurrence to the set of criticalities and/or discontinuities, which we assume are of a certain non-degenerate type, we can take advantage of the method of proof to obtain the same conclusion in a setting where the fibers can be higher dimensional manifolds.

Let us assume that $\varphi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \times \mathbb{Y}$ has the same skew-product form as before, but now:

(H₅) $f : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$ is a Borel measurable map such that $f_\theta : \{\theta\} \times \mathbb{Y} \rightarrow \mathbb{Y}$ is $C^{1+\alpha}$ away from a set of non-degenerate discontinuities \mathcal{D}_θ and/or criticalities \mathcal{C}_θ in the compact finite d -dimensional manifold \mathbb{Y} .

We fix a Riemannian metric on \mathbb{Y} , the corresponding distance function dist and norm $\|\cdot\|$ to be used in what follows. We also fix a normalized volume form Leb (Lebesgue measure) on \mathbb{Y} . The next regularity conditions on the derivatives will be needed.

(H₆) $f' : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d), (\theta, x) \mapsto Df_\theta(x)$ and $f'_1 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d), (\theta, x) \mapsto Df_\theta(x)^{-1}$ are Borel measurable maps with respect to the Borel σ -algebras of $\mathbb{X} \times \mathbb{Y}$ and $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$. In this last space we consider the topology induced by the usual operator norm $\|L\|_{\theta, x} := \sup\{\|L(v)\|_{f_\theta(x)} / \|v\|_x : \vec{0} \neq v \in T_x \mathbb{Y}\}$ for a linear map $L : T_x \mathbb{Y} \rightarrow T_{f_\theta(x)} \mathbb{Y}, (\theta, x) \in \mathbb{X} \times \mathbb{Y}$.

We also assume conditions (H₁) and (H₂) (or (H₂^{*})) and (H₃) on $\mathcal{S}, \mathcal{D}_\alpha$ and \mathcal{D}_F as before replacing I_0 by \mathbb{Y} throughout.

The non-degenerate assumption on the sets \mathcal{C}_θ and \mathcal{D}_θ mean that f_θ behaves like a power of the distance near the set of criticalities/discontinuities. More precisely: there are constants $B > 1$ and $\beta > 0$ for which, writing \mathcal{S}_θ for $\mathcal{S} \cap (\{\theta\} \times \mathbb{Y})$

$$(S1) \quad \frac{1}{B} \text{dist}(x, \mathcal{S}_\theta)^\beta \leq \frac{\|Df_\theta(x)v\|}{\|v\|} \leq B \text{dist}(x, \mathcal{S}_\theta)^{-\beta};$$

$$(S2) \quad \left| \log \|Df_\theta(x)^{-1}\| - \log \|Df_\theta(y)^{-1}\| \right| \leq B \frac{\text{dist}(x, y)}{\text{dist}(x, \mathcal{S}_\theta)^\beta};$$

$$(S3) \quad \left| \log |\det Df_\theta(x)^{-1}| - \log |\det Df_\theta(y)^{-1}| \right| \leq B \frac{\text{dist}(x, y)}{\text{dist}(x, \mathcal{S}_\theta)^\beta};$$

for every $\theta \in \mathbb{X}$ and $x, y \in \mathbb{Y} \setminus (\mathcal{S}_\theta)$ with $\text{dist}(x, y) < \text{dist}(x, \mathcal{S}_\theta)/2$ and $v \in T_x \mathbb{Y}$.

Given $\delta > 0$ we define the δ -truncated distance from $x \in \mathbb{Y}$ to \mathcal{S}_θ

$$\text{dist}_\delta(x, \mathcal{S}_\theta) = \begin{cases} 1 & \text{if } \text{dist}(x, \mathcal{S}_\theta) \geq \delta, \\ \text{dist}(x, \mathcal{S}_\theta) & \text{otherwise.} \end{cases}$$

We say that φ is *non-uniformly expanding along the fibers* if

- φ has positive Lyapunov exponent along the vertical direction according to $\nu \times \text{Leb}$:

$$(1.3) \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\alpha^j(\theta)}(f_{\theta}^j(x))^{-1}\| < 0, \quad \nu \times \text{Leb} - \text{a.e. } (\theta, x);$$

- φ has slow recurrence to the set of criticalities and discontinuities: for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$(1.4) \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_{\delta}(f_{\theta}^j(x), \mathcal{S}_{\alpha^j(\theta)}) < \epsilon, \quad \nu \times \text{Leb} - \text{a.e. } (\theta, x)$$

(the reader can recall the definition of \mathcal{S} in the statement of condition (H_1)).

Our result in this setting reads as follows.

Theorem D. *Let $\varphi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \times \mathbb{Y}$ be a skew-product as above satisfying (H_1) , (H_2) , (H_3) , (H_5) and (H_6) . Assume that φ non-uniformly expanding along the fibers. Then we obtain the same conclusions as in Theorem A.*

1.2. Strategy of the proof and organization of the text. The basic idea is to define measures on the vertical foliation of the skew-product, depending on the starting vertical leaf $\{\theta\} \times I_0$ or $\{\theta\} \times \mathbb{Y}$; show that these measures depend measurably on $\theta \in \mathbb{X}$ and can be integrated with respect to ν ; and then show that weak* accumulation points of these integrated measures are φ -invariant.

The assumption of positive Lyapunov exponent along the vertical direction, or the assumption of non-uniform expansion along the fibers, enables us to control the densities of these measures along the vertical direction on a certain subset of points which has “positive mass at infinity”. This provides us with an absolutely continuous component for every weak* accumulation point obtained before. Finally, using the uniqueness of Lebesgue decomposition and the smoothness assumption on f_{θ} allows us to obtain an invariant probability measure μ for the skew-product φ which is absolutely continuous with respect to the product measure $\nu \times m$ of the invariant measure on the base and Lebesgue measure on the interval. The ergodicity is obtained as a consequence of the fact that the invariant sets, with positive $\nu \times m$ -measure, have $\nu \times m$ -measure bounded away from zero.

In the next Section 2 we present some examples of application our main results. In Section 3 we construct the basic measures we will use to obtain the invariant probability measures for φ . In Section 4 we construct an absolutely continuous invariant probability measure for φ . In Section 6, we prove that the invariant sets with positive measure must to have measure bounded away from zero. As consequence of this result, we conclude the existence of ergodic absolutely continuous invariant probabilities. From these arguments it also follows the conclusion of Theorem A and Corollary B. In Section 7 we prove Theorem C, about existence of finitely many SRB probabilities for random non-uniformly expanding maps.

In Sections 3 and 4 we assume that the base dynamics $\alpha : \mathbb{X} \rightarrow \mathbb{X}$ is a bimeasurable bijection. We explain how to replace this condition by (H_2) in Section 5. Finally, in Section 8 we outline the arguments proving the main theorems in the setting with higher-dimensional fibers; and in Appendix A we prove the measurability of the sets used in the construction of the measures in the previous sections.

2. SOME EXAMPLES

Example 1. Skew-products of quadratic maps have been extensively studied. In [32, 11] is proved (1.2), with ν being Lebesgue measure on \mathbb{S}^1 , for the maps

$$F : \mathbb{S}^1 \times \mathbb{R} \rightarrow \mathbb{S}^1 \times \mathbb{R}, (\theta, x) \mapsto (k \cdot \theta, a_0 - x^2 + a \sin(2\pi\theta))$$

where $k \in \mathbb{Z}^+ \setminus \{1\}$ and $a_0 \in (1, 2]$ is such that 0 is preperiodic for the map $f_{a_0}(x) = a_0 - x^2$. In [28] the same map F as above was studied but with k a real parameter in the interval $(R_0, +\infty)$, where $1 < R_0 < 2$ was shown to exist so that, the map F with $k > R_0$ satisfies (1.2).

In [29] were considered skew-products $G(\theta, x) = (f_{a_1}^k(\theta), f_{a_0}(x) + \alpha s(\theta))$, where $f_a(x) := a - x^2$ and a_0, a_1 are parameters in the interval $(1, 2]$ such that the critical point is pre-periodic but not periodic, and $s : \mathbb{S}^1 \rightarrow [-1, 1]$ is a piecewise C^1 map. It was proved that there exist $k_0 \in \mathbb{Z}^+$ and a C^1 map s such that, for every small enough $\alpha > 0$ and all integers $k \geq k_0$, the map G satisfies (1.2), with $X = [f_{a_1}^2(0), f_{a_1}(0)]$ and ν being Lebesgue measure on the invariant interval X .

Note that the base transformation for the maps in [32, 11, 28] is (piecewise) expanding. For the maps in [29], it is non-uniformly expanding with critical points.

The existence of absolutely continuous invariant probability measures for all these maps is an immediate consequence of Theorem A.

Let us mention that the construction of the absolutely continuous invariant probability was obtained in [3] for the maps considered on [32, 11]. In [28] this conclusion was only achieved for a full Lebesgue measure subset of $(R_0, +\infty)$. The author in [29] did not obtain absolutely continuous invariant measures. Recently, in [2] was obtained the result for all the maps in [28, 29], as a byproduct of the application of inducing to study decay of correlations for the unique absolutely continuous invariant probability measure.

Example 2. We can produce examples where the base dynamics is essentially arbitrary. Let \mathbb{X} be the circle \mathbb{S}^1 and $\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ a measurable map preserving an ergodic probability measure ν . Let $\theta \mapsto f_\theta$ be a continuous family of maps of the interval $I_0 = [0, 1]$ such that

- for all $\theta \in \mathbb{S}^1$ the map $f_\theta : I_0 \rightarrow I_0$ is 2-to-1, with two branches $f_\theta|_{[0, 1/2]} : [0, 1/2] \rightarrow [0, 1]$ and $f_\theta|_{[1/2, 1]} : [1/2, 1] \rightarrow [0, 1]$ both increasing diffeomorphisms;
- on an arc A of \mathbb{S}^1 with $\nu(A) \geq 1 - \epsilon$ for some small $\epsilon > 0$ we have
 - for $\theta \in A$ the map f_θ is expanding; there exists $\sigma > 1$ such that $|Df_\theta(x)| \geq \sigma$ for all $x \in I_0$;
 - for $\theta \in \mathbb{S}^1 \setminus A$ the map f_θ does not contract too much: there exists $\delta > 0$ small such that $|Df_\theta(x)| \geq 1 - \delta$ for all $x \in I_0$.

In this setting we have that for $(\nu \times m)$ -a.e. (θ, x) , applying the Ergodic Theorem to the sequence $(\alpha^j(\theta))_{j \geq 0}$

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |Df_{\alpha^j(\theta)}(f_\theta^j(x))| &\geq \nu(A) \log \sigma + \nu(\mathbb{S}^1 \setminus A) \log(1 - \delta) \\ &\geq (1 - \epsilon) \log \sigma - \delta \epsilon > 0, \end{aligned}$$

where m is the Lebesgue measure on I_0 .

For a concrete expression we may take (see Figure 1)

$$(2.1) \quad f_t(x) = \begin{cases} tx + 2^\alpha(2-t)x^{1+\alpha} & \text{if } x \in [0, \frac{1}{2}) \\ 1 - t(1-x) - 2^\alpha(2-t)(1-x)^{1+\alpha} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

with $\alpha \in (0, 1)$ and $t \in (1/2, 3/2)$. We then take a function $t : \mathbb{S}^1 \rightarrow (1/2, 3/2)$ such that, for some small $a > 0$, satisfies $t(A) \subset (1+a, 3/2)$ and $t(\mathbb{S}^1 \setminus A) \subset (1-a, 1+a]$. Finally we define $\varphi(\theta, x) = (\alpha(\theta), f_{t(\theta)}(x))$.

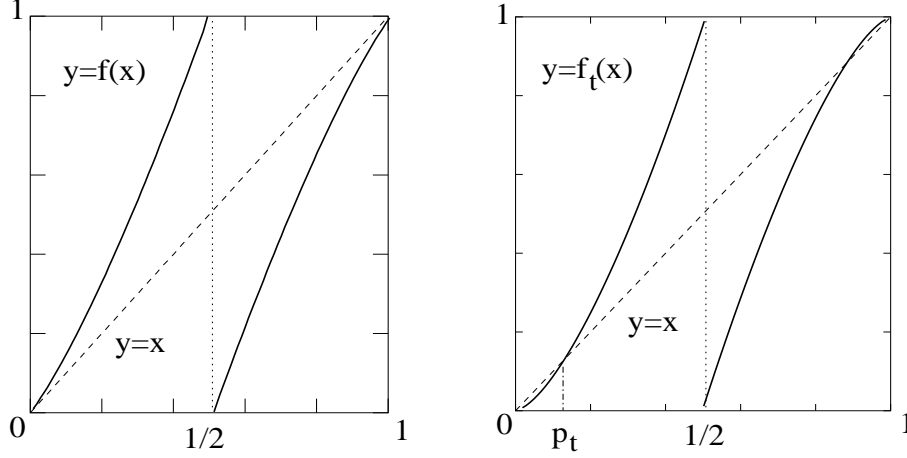
We remark that $\mathcal{D} = \mathbb{S}^1 \times \{1/2\}$ is such that every sequence z_k converging to \mathcal{D} on $\mathbb{S}^1 \times I_0$ is sent to a sequence $\varphi(z_k)$ whose accumulation points are contained in $\mathbb{S}^1 \times \{0, 1\}$, which is a forward invariant subset of φ . This implies the strong non-recurrence condition in (H_4^*) .

From Theorem A we have that φ admits a φ -invariant probability measure μ absolutely continuous with respect to $\nu \times m$.

Example 3. We can construct this example with α a circle diffeomorphism with irrational rotation number and ν an ergodic α -invariant probability which is non-atomic and singular with respect to m ; see e.g. [18, Theorem 12.5.1]. We note that in this way we have a *base map α with no average expansion*.

Example 4. We can adapt the construction in Example 2 with fibers of arbitrary dimension. We fix $k > 1$ in what follows.

Let again \mathbb{X} be the circle \mathbb{S}^1 and $\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ a measurable map preserving an ergodic probability measure ν . Let now $\theta \mapsto f_\theta$ be a continuous family of maps of the k -torus \mathbb{T}^k such that, as before,

FIGURE 1. The map f_1 (left) and the map f_t for $t < 1$ (right).

- on an arc A of \mathbb{S}^1 with $\nu(A) \geq 1 - \epsilon$ for some small $\epsilon > 0$ and some Riemannian norm $\|\cdot\|$ on \mathbb{T}^k we have:
 - for $\theta \in A$ the map f_θ is expanding: there exists $\sigma > 1$ such that $\|Df_\theta(x)^{-1}\| \leq 1/\sigma$ for all $x \in \mathbb{T}^k$;
 - for $\theta \in \mathbb{S}^1 \setminus A$ the map f_θ does not contract too much: there exists $\delta > 0$ small such that $\|Df_\theta(x)^{-1}\| \leq 1 + \delta$ for all $x \in \mathbb{T}^k$.

As before, in this setting, we have for $(\nu \times \text{Leb})$ -a.e. (θ, x) that, applying the Ergodic Theorem to the sequence $(\alpha^j(\theta))_{j \geq 0}$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\alpha^j(\theta)}(f_\theta^j(x))^{-1}\| \leq \nu(A) \log \sigma + \nu(\mathbb{S}^1 \setminus A) \log(1 + \delta) \\ \leq (1 - \epsilon) \log \sigma + \delta \epsilon < 0,$$

where Leb is the some volume from (Lebesgue measure) on \mathbb{T}^k . Since there are no criticalities or discontinuities, this shows that $\varphi(\theta, x) = (\alpha(\theta), f(\theta, x))$ is a non-uniformly expanding map along the fibers and we may apply Theorem D to conclude the existence of a probability measure μ absolutely continuous with respect to $\nu \times \text{Leb}$.

Example 5. Now we adapt the previous Example 4 to have a discontinuous family of fiber maps. We repeat the construction, keeping the choice of f_θ for $\theta \in A$ but replacing f_θ by the identity map on the torus for $\theta \in \mathbb{S}^1 \setminus A$.

We still have non-uniform expansion and we note that the discontinuities of the map F are on the boundary ∂A of the arc A of the circle, which is formed by a two points on the circle. Hence condition (H_3) is satisfied. We apply Theorem D to obtain a φ -invariant probability η absolutely continuous with respect to $\nu \times \text{Leb}$.

Example 6. We present an example of a C^∞ map T away from a denumerable singular set, which is non-uniformly expanding and has infinitely many ergodic absolutely continuous invariant probability measures.

On the one hand, considering $\alpha = T$ as the base map and a constant fiber map $f(x) = 4x(1 - x)$ of the interval which has positive Lyapunov exponents for Lebesgue almost all point, a unique critical point and negative Schwarzian derivative, we obtain a direct product $\varphi = \alpha \times f$. The map f admits a unique ergodic absolutely continuous invariant probability measure μ . Thus we can apply our arguments to each ergodic absolutely continuous invariant probability measure ν_k for α to obtain $\nu_k \times \mu$ as an ergodic absolutely continuous invariant probability measure for φ . In this way φ has a countable set of distinct absolutely continuous invariant probability measures.

On the other hand, considering the direct product $\varphi = \alpha \times T$ of any map α of a metric space with an ergodic probability measure ν , with T on the fibers, we obtain an example with infinitely many ergodic absolutely continuous invariant measures $\nu \times \nu_k$ with the same marginal ν .

The map T is easily described as the standard doubling map

$$f : x \in [0, 1] \mapsto \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

conveniently rescaled on the unit interval infinitely many times, as follows, see figure 2:

$$T(x) := \sum_{n \geq 1} \begin{cases} \frac{1}{2^n} + \frac{1}{2^n} f(2^n(x - 2^{-n})) & \text{if } x \in \left] \frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \\ 0 & \text{otherwise} \end{cases}.$$

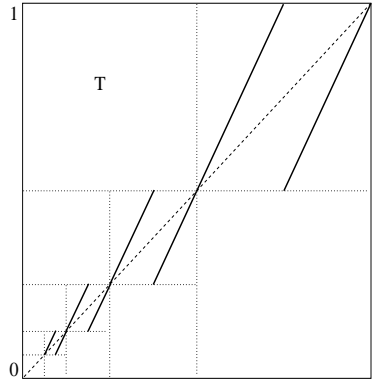


FIGURE 2. A sketch of the map T .

It is clear that $DT \equiv 2$ and $DT^2 \equiv 0$ outside the compact set $\mathcal{S} := \{0\} \cup \{2^{-n}, 2^{-n} + 2^{-(n+1)} : n \in \mathbb{Z}^+\}$. It is easy to see that Lebesgue measure m on $[0, 1]$ is invariant and each interval $[2^{-n}, 2^{-n+1}]$ supports an ergodic component of m given by the normalized restriction of m to this interval.

Moreover it is straightforward to check that the set \mathcal{S} satisfies conditions (S1) through (S3) with constants $B = \beta = 1$, so \mathcal{S} is a non-degenerate singular set for T . In addition, conditions (H_2) , (H_3) and (H_4^*) are also easily checked.

However the slow recurrence condition is not satisfied: for each given $\delta > 0$ and $N > 1$ there exists $k > N$ such that $2^{-k+1} < \delta$ and we have

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_\delta(T^j(x), \mathcal{S}) \geq k > N \quad \text{for all } x \in (2^{-k}, 2^{-k+1}).$$

But this condition fails in a small set: for each $N > 1$ the points for which the above inequality holds are contained in $[0, 2^{-[\log_2 N]+1})$, where $[x]$ denotes the integer part of x .

3. BASIC INVARIANT MEASURES

We assume from now on that the skew-product map satisfies (H_1) , (H_2^*) , (H_3) and (H_4) (or (H_4^*)) or we replace (H_4) by non-uniform expansion, where

(H_2^*) $\alpha : \mathbb{X} \rightarrow \mathbb{X}$ is a bimeasurable bijection with an ergodic invariant probability measure ν such that $\nu(\mathcal{D}_\alpha) = 0$, where \mathcal{D}_α is the set of discontinuity points of α .

In section 5 we show how to replace condition (H_2^*) by (H_2) .

Let us denote by m the Lebesgue measure on I_0 . Since α is invertible, the functions $f_{\alpha^{-j}(\theta)}^j$ are well defined and they send $\{\alpha^{-j}(\theta)\} \times I_0$ on $\{\theta\} \times I_0$, for $\theta \in \mathbb{X}$, $j \geq 1$. Thus, we can define the following

measures on I_0 , for every $\theta \in \mathbb{X}$ and every $n \in \mathbb{N}$,

$$\eta_n(\theta) = \frac{1}{n} \sum_{j=1}^n (f_{\alpha^{-j}(\theta)}^j)_* \mathbf{m}$$

and using them, for every $n \in \mathbb{N}$ we define the following measures on $\mathbb{X} \times I_0$,

$$\eta_n = \int \eta_n(\theta) d\nu(\theta).$$

The integral above means that for any continuous function $g : \mathbb{X} \times I_0 \rightarrow \mathbb{R}$ we have

$$\eta_n(g) = \int g d\eta_n = \int \left(\int g(\theta, x) d\eta_n(\theta)(x) \right) d\nu(\theta).$$

Let us denote by $\mathcal{B}_{\mathbb{X}}$ the Borel σ -algebra on \mathbb{X} . To be able to define the measure η_n , we need that for every continuous function $h : I_0 \rightarrow \mathbb{R}$ the map

$$\theta \mapsto \eta_n(\theta)(h) = \int h d\eta_n(\theta)$$

is measurable. This is proved in Appendix A.

Assuming that these measures are all well-defined, we can easily prove some key properties of the accumulation points of $(\eta_n)_{n \geq 1}$.

Lemma 3.1. *For every probability measure η which is a weak* limit of $(\eta_n)_{n \geq 1}$ we have that $\eta_n(A \times I_0) = \nu(A)$ for each $n \geq 1$ and $\eta(A \times I_0) = \nu(A)$, for all $A \in \mathcal{B}_{\mathbb{X}}$.*

Proof. We fix A and η as in the statement. Then we have for all $n \in \mathbb{Z}^+$ by definition $\eta_n(A \times I_0) = \int_A \eta_n(\theta)(I_0) d\nu(\theta) = \nu(A)$. If we take $A \in \mathcal{B}_{\mathbb{X}}$ such that $\eta(\partial(A \times I_0)) = \eta((\partial A) \times I_0) = 0$, then using $\eta_{n_k} \xrightarrow[k \rightarrow +\infty]{w^*} \eta$ we get $\eta(A \times I_0) = \nu(A)$. Since the family of these sets generates $\mathcal{B}_{\mathbb{X}}$ modulo η -null sets, we are done. \square

Lemma 3.2. *For every probability measure η which is a weak* limit of $(\eta_n)_{n \geq 1}$ we have that $\eta(\mathcal{D}) = 0$, where \mathcal{D} is the set of discontinuity points of φ .*

Proof. We consider the following cases.

Case 1: the maps f_θ are C^3 for all $\theta \in \mathbb{X}$, that is, there are no discontinuities along the vertical direction: $\mathcal{D}_\theta = \emptyset$ for all $\theta \in \mathbb{X}$. Thus, it holds that $\mathcal{D} \subset (\mathcal{D}_\alpha \times I_0) \cup (\mathcal{D}_F \times I_0)$. Then we have, by Lemma 3.1, that $\eta(\mathcal{D}) \leq \eta(\mathcal{D}_\alpha \times I_0) + \eta(\mathcal{D}_F \times I_0) \leq \nu(\mathcal{D}_\alpha) + \nu(\mathcal{D}_F) = 0$ by (H_2^*) and (H_3) .

Case 2: we have discontinuities $\mathcal{D}_\theta \neq \emptyset$ for some $\theta \in \mathbb{X}$. But we assume that there is no recurrence to the set $\mathcal{D} = \{(\theta, x) : x \in \mathcal{D}_\theta, \theta \in \mathbb{X}\}$; Section 1.1. see condition (H_4^*) . Hence for every given $\ell \in \mathbb{Z}^+$ we can find an open neighborhood $V = V_\ell$ of \mathcal{D} in $\mathbb{X} \times I_0$ such that $\varphi^k(V) \cap V = \emptyset$ for all $k = 1, \dots, \ell$. This implies that for any $z \in \mathbb{X} \times I_0$ we have $\sum_{j=1}^n \chi_{V_\ell}(\varphi^j(z)) \leq (n/\ell) + 1$. Thus, since $\eta(V) \leq \liminf_{n \rightarrow +\infty} \eta_n(V)$ (see e.g. [8, Theorem 2.1]), it is enough to estimate for every big enough $n \in \mathbb{Z}^+$, using that ν is α -invariant and that α is invertible

$$\begin{aligned} \eta_n(V) &= \int \int \frac{1}{n} \sum_{j=1}^n \chi_V(\theta, f_{\alpha^{-j}(\theta)}^j(x)) d\mathbf{m}(x) d\nu(\theta) \\ &= \int \frac{1}{n} \int \sum_{j=1}^n \chi_V(\varphi^j(\alpha^{-j}(\theta), x)) d\mathbf{m}(x) d\nu(\theta) \\ &= \int \frac{1}{n} \int \sum_{j=1}^n \chi_V(\varphi^j(\theta, x)) d\mathbf{m}(x) d\nu(\theta) \leq \frac{2}{\ell}. \end{aligned}$$

So for every $\ell > 1$ we can find an open neighborhood V of \mathcal{D} such that $\eta(\mathcal{D}) \leq \eta(V) \leq 2/\ell$. Finally, since $\mathcal{D} \subseteq (\mathcal{D}_\alpha \times I_0) \cup (\mathcal{D}_F \times I_0) \cup \mathcal{D}$ we obtain from the above together with Lemma 3.1

$$\eta(\mathcal{D}) \leq \eta(\mathcal{D}_\alpha \times I_0) + \eta(\mathcal{D}_F \times I_0) + \eta(\mathcal{D}) = \nu(\mathcal{D}_\alpha) + \nu(\mathcal{D}_F) = 0$$

as stated.

Case 3: In the higher dimensional setting, we have slow recurrence to the set of discontinuities $\mathcal{D} \subset \mathcal{S}$ of φ in the vertical direction. Arguing by contradiction, let us assume that $\eta(\mathcal{D}) > 0$. Then there exists $a > 0$ such that $\eta(B(\mathcal{D}, \varrho)) > a$ for all $\varrho > 0$.

We fix $0 < \varepsilon < a$ and then find $\delta > 0$ given by the slow recurrence condition (1.4). After that we fix $0 < \varrho < \delta$ so that

$$\inf\{-\log \text{dist}((\theta, x), \mathcal{D}) : (\theta, x) \in B(\mathcal{D}, \varrho)\} > 1 \quad \text{and} \quad \eta_n(\partial B(\mathcal{D}, \varrho)) = 0, \quad n \geq 1$$

and also $\eta(\partial B(\mathcal{D}, \varrho)) = 0$. Then we note that, for each $n \geq 1$, since ν is α -invariant

$$\begin{aligned} a < \eta_n(B(\mathcal{D}, \varrho)) &= \int \int \frac{1}{n} \sum_{j=1}^n \chi_{B(\mathcal{D}, \varrho)}(\theta, f_{\alpha^{-j}(\theta)}^j(x)) d\text{Leb}(x) d\nu(\theta) \\ &= \int \frac{1}{n} \int \sum_{j=1}^n \chi_{B(\mathcal{D}, \varrho)}(\varphi^j(\alpha^{-j}(\theta), x)) d\text{Leb}(x) d\nu(\theta) \\ &= \int \frac{1}{n} \int \sum_{j=1}^n \chi_{B(\mathcal{D}, \varrho)}(\varphi^j(\theta, x)) d\text{Leb}(x) d\nu(\theta) \\ &\leq \int \int \frac{1}{n} \sum_{j=1}^n -\log \text{dist}_\delta(\varphi^j(\theta, x), \mathcal{D}) d\text{Leb}(x) d\nu(\theta). \end{aligned}$$

Moreover, for big enough n we get $\varepsilon > \eta_n(B(\mathcal{D}, \varrho)) \geq a/2$ thus $a > 2\varepsilon$. This contradiction concludes the proof, since $\mathcal{D} \subseteq (\mathcal{D}_\alpha \times I_0) \cup (\mathcal{D}_F \times I_0) \cup \mathcal{D}$ as in Case 2. \square

Lemma 3.3. *Every weak* limit of $(\eta_n)_{n \geq 1}$ is a φ -invariant probability measure.*

Proof. Let us suppose, without loss of generality, that the sequence converges in the weak* topology to some probability measure, i.e., $\eta_n \rightarrow \eta$ when $n \rightarrow \infty$. See Lemma 4.6 and Remark 4.7.

Let $g : \mathbb{X} \times I_0 \rightarrow \mathbb{R}$ be a continuous and bounded function. We note that $\eta_n(g \circ \varphi)$ can be rewritten as

$$\begin{aligned} \iint g(\varphi(\theta, x)) d\eta_n(\theta)(x) d\nu(\theta) &= \iint g(\alpha(\theta), f_\theta(x)) d\eta_n(\theta)(x) d\nu(\theta) \\ &= \int \left(\frac{1}{n} \sum_{j=1}^n (f_{\alpha^{-1}(\theta)} \circ \dots \circ f_{\alpha^{-j}(\theta)})_* \mathbf{m} \right) (g \circ \varphi) d\nu(\theta) \\ &= \frac{1}{n} \sum_{j=1}^n \iint g(\alpha(\theta), f_\theta(f_{\alpha^{-1}(\theta)} \circ \dots \circ f_{\alpha^{-j}(\theta)}(x))) d\mathbf{m}(x) d\nu(\theta). \end{aligned}$$

But the last integral equals

$$\int \frac{1}{n} \left(\int \sum_{j=1}^{n+1} g(\alpha(\theta), (f_\theta \circ f_{\alpha^{-1}(\theta)} \circ \dots \circ f_{\alpha^{-j+1}(\theta)})(x)) d\mathbf{m}(x) - \int g(\alpha(\theta), f_\theta(x)) d\mathbf{m}(x) \right) d\nu(\theta)$$

that is $\int \left(\frac{n+1}{n} \eta_{n+1}(\alpha(\theta))(g) - \frac{1}{n} ((f_\theta)_* \mathbf{m})(g(\alpha(\theta), \cdot)) \right) d\nu(\theta)$. We note that the last integral is bounded by $\sup |g|$, which is finite.

Now since $\eta(\mathcal{D}) = 0$ by Lemma 3.2, we then arrive at (see e.g. [8, Theorem 2.7])

$$(\varphi_*\eta)g = \lim_{n \rightarrow \infty} \eta_n(g \circ \varphi) = \lim_{n \rightarrow \infty} \int \frac{n+1}{n} \eta_{n+1}(\alpha(\theta))(g) d\nu(\theta).$$

But ν is α -invariant and the function $\theta \mapsto \eta_{n+1}(\alpha(\theta))(g)$ is measurable, hence the last expression equals

$$\lim_{n \rightarrow \infty} \int \frac{n+1}{n} \eta_{n+1}(\theta)(g) d\nu(\theta) = \lim_{n \rightarrow \infty} \frac{n+1}{n} \eta_{n+1}(g) = \eta(g).$$

This concludes the proof. \square

4. ABSOLUTELY CONTINUOUS INVARIANT MEASURES

Now we are going to define measures which are absolutely continuous along the vertical fibers. For this, we will use the notion of hyperbolic-like times used in [30].

4.1. Notations and main technical result. We state a result for sequences of one dimensional maps. This result is used to analyze the dynamics of the skew-product restricted to the vertical leaves. Since we have to consider skew-products in the different settings (H_4) and (H_4^*) , we also need to state the result for sequences of one dimensional maps with conditions given by these two settings. For $k \geq 0$, let us denote by \mathcal{C}_k and \mathcal{D}_k the set of critical points and the set of discontinuities, respectively, of $f_k : I_0 \rightarrow I_0$.

We say that:

- a sequence of one dimensional maps $\{f_k\}$ satisfies (\widetilde{H}_4) if: f_k are C^1 maps, $p := \sup\{\#\mathcal{C}_k, k \in \mathbb{N}\} < \infty$ and $\Gamma := \sup\{|f'_k(x)|, k \in \mathbb{N}, x \in I_0\} < \infty$ and the sequence $\{f'_k\}$ is equicontinuous.
- a sequence of one dimensional maps $\{f_k\}$ satisfies (\widetilde{H}_4^*) if: f_k is a map such that restricted to each connected component of $I_0 \setminus \mathcal{D}_k$, is a C^1 diffeomorphism onto its image, $p := \sup\{\#\mathcal{D}_k, k \in \mathbb{N}\} < \infty$ and $\Gamma := \sup\{|f'_k(x)|, k \in \mathbb{N}, x \notin \mathcal{D}_k\} < \infty$.

Finally we assume that for every $\ell \in \mathbb{Z}^+$, there exist $\epsilon > 0$ and neighborhoods $V_\epsilon \mathcal{D}_k$ of \mathcal{D}_k (for all $k \geq 0$) such that

$$(4.1) \quad f_i^j(V_\epsilon \mathcal{D}_i) \cap V_\epsilon \mathcal{D}_{i+j} = \emptyset \quad \text{for } i \geq 0, 0 \leq j \leq \ell.$$

where $f_i^j = f_{i+j-1} \circ \dots \circ f_{i+1} \circ f_i$.

Let us recall some additional definitions (see [30] for more details). For every $x \in I_0$, $i \in \mathbb{N}$, we denote

$$f^i(x) := f_{i-1} \circ \dots \circ f_1 \circ f_0(x),$$

and we write $T_i(\{f_k\}, x)$ for the maximal interval $T \subset I_0$, containing x such that $f_{|T}^i$ is a C^3 diffeomorphism, and $r_i(\{f_k\}, x)$ for the minimum between the lengths of the connected components of $f^i(T_i(\{f_k\}, x) \setminus \{x\})$.

The following is a central technical result in our arguments. For the proof see subsection 4.3.

Theorem 4.1. *Let $\{f_k\}$ be a sequence of maps $f_k : I_0 \rightarrow I_0$ which satisfies (\widetilde{H}_4) (or (\widetilde{H}_4^*)). Assume that for some $\lambda > 0$,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x)| > 2\lambda$$

for every $x \in E \subset I_0$. Then, there exists $\varsigma > 0$ such that

$$(4.2) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n r_i(\{f_k\}, x) \geq 3\varsigma$$

Lebesgue almost every $x \in E$. Moreover, in the case of $\{f_k\}$ satisfy condition (\widetilde{H}_4) , ς depends only on λ , the modulus of continuity and the uniform bound for the derivatives of the sequence $\{f_k\}$, and in the uniform bound p for the number of critical points. In the case of $\{f_k\}$ satisfy condition (\widetilde{H}_4^) , ς depends only on λ , the uniform bound for the derivatives of the sequence $\{f_k\}$ (outside of discontinuities), the uniform bound p for the number of discontinuity points and the uniformity of ϵ on condition (4.1).*

For our purposes the following sets are very useful:

$$\begin{aligned}\mathcal{H}_i(\{f_k\}, \sigma) &= \{x \in I_0; r_i(\{f_k\}, x) > \sigma\}; \\ H_i(\{f_k\}, \sigma) &= \left\{x \in I_0; r_i(\{f_k\}, x) > \sigma \text{ and } |f^i(T_i(\{f_k\}, x))| > 3\sigma\right\}.\end{aligned}$$

We will prove below that every connected component of $H_i(\{f_k\}, \sigma)$ is sent diffeomorphically by f^j onto its image with bounded distortion and the Lebesgue measure of the image is bounded away from zero. We are interested in applying the last theorem to every sequence $\{f_{\alpha^j(\theta)}\}_{j \in \mathbb{Z}^+}$, for each $\theta \in \mathbb{X}$. For simplicity, from now on, for $i \in \mathbb{N}$, $r_i(\theta, x)$ denotes the set $r_i(\{f_k\}, x)$, where $f_k = f_{\alpha^k(\theta)}$ for every $k \in \mathbb{N}$, $\theta \in \mathbb{X}$. Analogously for the sets $T_i(\theta, x)$, $\mathcal{H}_i(\theta, \sigma)$ and $H_i(\theta, \sigma)$.

We need the following result showing that (\widetilde{H}_4^*) is a consequence of (H_4^*) .

Lemma 4.2. *The above condition (4.1) is a consequence of the assumption (H_4^*) .*

Proof. We fix $\ell \in \mathbb{Z}^+$ and V given by (H_4^*) . Consider also $i \geq 0$ and $0 \leq j \leq \ell$. We note that, by the skew-product nature of φ

$$\varphi^j(V \cap ((\{\alpha^i(\theta)\} \times I_0)) \subset (\{\alpha^{i+j}(\theta)\} \times I_0) \cap \varphi^j(V).$$

We now observe that the intersection in (4.1) equals

$$\pi_2(\varphi^j(V \cap (\{\alpha^i(\theta)\} \times I_0)) \cap (V \cap (\{\alpha^{i+j}(\theta)\} \times I_0))) \subset \pi_2((\{\alpha^{i+j}(\theta)\} \times I_0) \cap \varphi^j(V \cap V)) = \emptyset,$$

where $\pi_2 : \mathbb{X} \times I_0 \rightarrow I_0$ is the projection on the second coordinate. So we can use the neighborhoods V given by (H_4^*) to obtain the neighborhoods $V_\epsilon \mathcal{D}_i$ in (4.1). \square

Remark 4.3. The fact that $\epsilon > 0$ does not depend on the sequence of maps chosen relies on the choice in (H_4^*) of the neighborhood V of the closure $\overline{\mathcal{D}}$ of the set of discontinuities in $\mathbb{X} \times I_0$.

We need the following result in the rest of the arguments.

Lemma 4.4 (Pliss). *Given $A \geq c_2 > c_1 > 0$, let $\xi = (c_2 - c_1)/(A - c_1)$. Then, given any real numbers a_1, \dots, a_N such that*

$$\sum_{j=1}^N a_j \geq c_2 N \quad \text{and} \quad a_j \leq A \text{ for every } 1 \leq j \leq N,$$

there are $l > \xi N$ and $1 < n_1 < \dots < n_l \leq N$ so that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n) \quad \text{for every } 0 \leq n < n_i \text{ and } i = 1, \dots, l.$$

Proof. See [22, Lemma 11.8]. \square

Thus, by the last theorem and using the Lemma of Pliss, we have the following.

Corollary 4.5. *Let $\varphi : \mathbb{X} \times I_0 \rightarrow \mathbb{X} \times I_0$ be a skew-product as above satisfying (H_1) and (H_4) (or (H_1) and (H_4^*)). Assume that there exists a set $E \subset \mathbb{X} \times I_0$ and $\lambda > 0$ such that*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df_\theta^n(x)| > 2\lambda$$

for every $(\theta, x) \in E$ and let us denote by $E(\theta)$ the θ -section of the set E , that is, $E(\theta) = \{x \in I_0 : (\theta, x) \in E\}$. Then there exist $\varsigma > 0$ and $\xi > 0$ such that for n big enough do not depend on θ .

$$\int \frac{1}{n} \sum_{i=1}^n m(\mathcal{H}_i(\theta, \varsigma) \cap E(\theta)) \, dv(\theta) \geq \frac{\xi}{2} (v \times m)(E).$$

Proof. Let us fix $\theta \in \mathbb{X}$ and consider the sequence $\{f_{\alpha^j(\theta)}\}_{j \in \mathbb{Z}^+}$. Let $\varsigma > 0$ be the constant found on Theorem 4.1. We consider the measure π_n in $\{1, 2, \dots, n\}$ defined by $\pi_n(B) = \#(B)/n$, for every subset B . Using Fubini's theorem, we have

$$\frac{1}{n} \sum_{i=1}^n m(\mathcal{H}_i(\theta, \varsigma) \cap E(\theta)) = \int \int_{I_0} \chi(x, i) d m(x) d \pi_n(i) = \int_{I_0} \int \chi(x, i) d \pi_n(i) d m(x)$$

where $\chi(x, i) = 1$ if $x \in \mathcal{H}_i(\theta, \varsigma) \cap E(\theta)$ and $\chi(x, i) = 0$ otherwise. Applying Pliss Lemma 4.4, we conclude the existence of $\xi > 0$ such that $\int \chi(x, i) d \pi_n(i) \geq \xi$ if x is such that $x \in E(\theta)$ and $\sum_{i=1}^n r_i(\theta, x) \geq 2\varsigma n$. Hence

$$\frac{1}{n} \sum_{i=1}^n m(\mathcal{H}_i(\theta, \varsigma) \cap E(\theta)) \geq \xi m \left(\left\{ x \in E(\theta); \sum_{i=1}^n r_i(\theta, x) \geq 2\varsigma n \right\} \right).$$

By Theorem 4.1, we have that

$$m \left(\left\{ x \in E(\theta) : \sum_{i=1}^n r_i(\theta, x) \geq 2\varsigma n, \text{ for all } n \geq N \right\} \right) \rightarrow m(E(\theta))$$

when $N \rightarrow \infty$. Since the constant ς is the same for any sequence $\{f_{\alpha^j(\theta)}\}_{j \in \mathbb{Z}^+}$, varying $\theta \in \mathbb{X}$, the result follows using the Dominated Convergence Theorem. \square

For any $\sigma > 0$, if $r_i(\{f_k\}, x) > 2\sigma$ then $|f^i(T_i(\{f_k\}, x))| > 4\sigma$. Thus, $\mathcal{H}_i(\{f_k\}, 2\sigma) \subset H_i(\{f_k\}, \sigma) \subset \mathcal{H}_i(\{f_k\}, \sigma)$. Therefore, we get a similar result to the last corollary for H_i instead of \mathcal{H}_i .

4.2. Construction of absolutely continuous invariant probability measures. Assume that we are in the conditions of Theorem A. Clearly, the set Z in the statement of Theorem A and Corollary B may be taken positively invariant under φ . Given any $\lambda > 0$, let $Z(\lambda)$ be the set of points in Z for which the limit in (1.2) is greater than 2λ . Then $Z(\lambda)$ is positively invariant. Let us fix a constant $\lambda > 0$ such that $\nu \times m(Z(\lambda)) > 0$. Let $\varsigma > 0$ be the constant found on Theorem 4.1. As usual, $Z(\theta, \lambda)$ denotes the θ -section of the set $Z(\lambda)$. Thus, we define the following measures on I_0 , for every $n \in \mathbb{N}$ and $\theta \in \mathbb{X}$

$$(4.3) \quad \mu_n(\theta) = \frac{1}{n} \sum_{j=1}^n (f_{\alpha^{-j}(\theta)}^j)_* (m \mid Z(\alpha^{-j}(\theta), \lambda) \cap H_j(\alpha^{-j}(\theta), \varsigma)).$$

Using these measures, for every $n \in \mathbb{N}$ we define the following on $\mathbb{X} \times I_0$,

$$(4.4) \quad \mu_n = \int \mu_n(\theta) d\nu(\theta)$$

Again we need to show that for every continuous function $h : I_0 \rightarrow \mathbb{R}$ the map

$$\theta \mapsto \mu_n(\theta)(h) = \int h d\mu_n(\theta)$$

is measurable. This is proved in Appendix A.

Lemma 4.6. *For all $n \geq 1$ and $A \in \mathcal{B}_{\mathbb{X}}$ we have $\mu_n(A \times I_0) \leq \nu(A)$. Moreover, this conditions ensures that the sequence $(\mu_n)_{n \geq 1}$ of measures is tight in $\mathbb{X} \times I_0$; thus it is relatively compact in the weak* topology of measures in $\mathbb{X} \times I_0$.*

Proof. We just observe that $\mu_n(A \times I_0) = \int_A \mu_n(\theta)(I_0) d\nu(\theta)$ by definition, and also $\mu_n(\theta)(I_0) \leq 1$ for each θ . In addition, from this property and the assumption that μ_n are Borel measures on \mathbb{X} which is a separable metrizable and complete topological space, given $\epsilon > 0$ we can fix a compact subset $\mathbb{X}_0 \subset \mathbb{X}$ such that $\nu(\mathbb{X} \setminus \mathbb{X}_0) < \epsilon$ and we obtain

$$\mu_n(\mathbb{X} \times I_0 \setminus (\mathbb{X}_0 \times I_0)) = \mu_n((\mathbb{X} \setminus \mathbb{X}_0) \times I_0) \leq \nu(\mathbb{X} \setminus \mathbb{X}_0) < \epsilon$$

uniformly in $n \geq 1$, as required for tightness of the family $(\mu_n)_{n \geq 1}$. We can now apply Prokhorov's Theorem to obtain the final conclusion of the statement of the lemma; see [8, Chapter 1, Section 5]. \square

Remark 4.7. Lemma 3.1 together with the previous arguments also shows that $(\eta_n)_{n \geq 1}$ is a tight sequence of probability measures in $\mathbb{X} \times I_0$.

Now we obtain the absolute continuity of $\mu_n(\theta)$ with respect to m .

Lemma 4.8. *There exists $K > 0$ such that for any measurable subset $A \subset I_0$,*

$$\mu_n(\theta)(A) \leq K m(A)$$

for every $\theta \in \mathbb{X}, n \in \mathbb{N}$. Moreover, K depends only on the constant ς in the definition of $H_i(\theta, \varsigma)$.

Proof. For every connected component J of $H_j(\alpha^{-j}(\theta), \varsigma)$, $J \subset T$ and $f_{\alpha^{-j}(\theta)}^j$ restricted to T is a C^3 diffeomorphism. Moreover, there exists τ (depending only on ς) such that $f_{\alpha^{-j}(\theta)}^j(T)$ contains a τ -scaled neighborhood of $f_{\alpha^{-j}(\theta)}^j(J)$ (i.e., both connected components of $f_{\alpha^{-j}(\theta)}^j(T) \setminus f_{\alpha^{-j}(\theta)}^j(J)$ have length $\geq \tau |f_{\alpha^{-j}(\theta)}^j(J)|$). By Koebe Principle, $f_{\alpha^{-j}(\theta)}^j$ restricted to J has bounded distortion (by a constant K' depending only on ς). Since $|f_{\alpha^{-j}(\theta)}^j(J)|$ is bounded away from zero, we conclude that $(f_{\alpha^{-j}(\theta)}^j)_*(m \mid Z(\alpha^{-j}(\theta), \lambda) \cap H_j(\alpha^{-j}(\theta), \varsigma))(A) \leq K m(A)$ for any measurable set $A \subset I_0$. \square

From the previous lemma we deduce the absolute continuity of μ_n with respect to $\nu \times m$.

Lemma 4.9. *There exists $K > 0$ such that for any $W \in \mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}$ we have $\mu_n(W) \leq K \cdot (\nu \times m)(W)$ for all $n \in \mathbb{N}$.*

Proof. The set $\mathcal{A} = \{W \in \mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}; \mu_n(W) \leq K \cdot (\nu \times m)(W)\}$ is a σ -algebra. On the other hand, if $W = F \times A$ for some $F \in \mathcal{B}_{\mathbb{X}}, A \in \mathcal{B}_{I_0}$, we conclude, from the definition of μ_n and the last claim, that $W \in \mathcal{A}$. This is enough to conclude the proof. \square

Now we extend the results of the previous lemmas to the cluster points of the sequence μ_n in the weak* topology.

Lemma 4.10. *There exists $K > 0$ such that, for any weak* limit μ of $\{\mu_n\}_n$, we have $\mu(W) \leq K \cdot (\nu \times m)(W)$ for any $W \in \mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}$.*

Proof. The set $\mathcal{A} = \{W \in \mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}; \mu(W) \leq K \cdot (\nu \times m)(W)\}$ is a σ -algebra. Since μ_{n_k} converges in the weak* topology to μ , $\mu(W) \leq \liminf \mu_{n_k}(W)$ for any open set W . Also note that from Lemma 4.9, for open sets $W \in \mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}$, $\mu_n(W) \leq K \cdot (\nu \times m)(W)$ for all $n \in \mathbb{N}$. As these sets generate $\mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}$, the claim follows. \square

By definition μ_n is a part of the measure η_n , for all $n \in \mathbb{N}$. Let ξ_n be a measure such that

$$(4.5) \quad \eta_n = \mu_n + \xi_n$$

for all $n \in \mathbb{N}$. From Lemma 4.6 and Remark 4.7 we assume, without loss of generality, that there exist some subsequence $\{n_k\}_k$ and measures η, μ, ξ such that $\eta_{n_k}, \mu_{n_k}, \xi_{n_k}$ converge to η, μ, ξ , when $k \rightarrow \infty$, respectively. We then have

$$(4.6) \quad \eta = \mu + \xi.$$

Let β_1 and β_2 be measures on the same measurable space. As usually, if β_1 is absolutely continuous with respect to β_2 , we write $\beta_1 \ll \beta_2$; and if β_1 is singular with respect to β_2 , we write $\beta_1 \perp \beta_2$.

Next we show that the Lebesgue decomposition of an invariant measure with respect to any finite measure λ , for a non-singular transformation, is formed by invariant measures.

Lemma 4.11. *Let us assume that a measurable transformation $T : (X, \mathcal{X}) \rightarrow (X, \mathcal{X})$ satisfies $T_*\lambda \ll \lambda$ for some finite measure λ in (X, \mathcal{X}) (that is, T is non-singular with respect to λ). We assume also that a T -invariant probability measure η is given with Lebesgue decomposition $\eta = \mu + \xi$, with $\mu \ll \lambda$ and $\xi \perp \lambda$. Then both μ and ξ are T -invariant measures.*

Proof. Since $\xi \perp \lambda$, we may find $E \in \mathcal{X}$ such that $\lambda(E) = 0$ and $\xi(X \setminus E) = 0$. In particular, $\xi(A) = \xi(A \cap E)$ for all $A \in \mathcal{X}$. Because $\lambda(E) = 0 = \lambda(T^{-1}(E))$ we get

$$\xi(T^{-1}(E)) = \mu(T^{-1}(E)) + \xi(T^{-1}(E)) = \eta(T^{-1}(E)) = \eta(E) = \mu(E) + \xi(E) = \xi(E)$$

and E is T -invariant $\xi \bmod 0$, i.e., $\xi(E \Delta T^{-1}(E)) = 0$. Hence $\xi(X \setminus T^{-1}(E)) = 0$ and $\lambda(T^{-1}(E)) = 0$. Thus for $A \in \mathcal{X}$

$$\xi(T^{-1}(A)) = \xi(T^{-1}(A) \cap T^{-1}(E)) = \xi(T^{-1}(A \cap E)) = \xi(A \cap E) = \xi(A)$$

since $\xi(T^{-1}(A \cap E)) = \eta(T^{-1}(A \cap E)) = \eta(A \cap E) = \xi(A \cap E)$. We have proved that ξ is T -invariant.

Therefore

$$\mu + \xi = \eta = T_*\eta = T_*\mu + T_*\xi = T_*\mu + \xi$$

shows that $T_*\mu = \mu$ and μ is also T -invariant □

4.2.1. Existence of absolutely continuous invariant probability measure. Now we use the previous results to complete the proof of existence of an absolutely continuous invariant probability measure for φ . The ergodicity is proved in Section 6.

Proposition 4.12. *Let $\varphi : \mathbb{X} \times I_0 \rightarrow \mathbb{X} \times I_0$ be a skew-product as above satisfying (H_1) , (H_2) , (H_3) and (H_4) (or (H_4^*)). Assume that $\nu \times m(Z(\lambda)) > 0$. Then there exists an absolutely continuous invariant measure which gives positive mass to $Z(\lambda)$.*

Proof. Let us consider the measures μ_n given by (4.4). We recall that by (4.5) and (4.6) we have $\eta = \mu + \xi$ with $\mu \ll \nu \times m$. By the Lebesgue Decomposition Theorem, there exist (unique) measures ξ^{ac} and ξ^s such that $\xi^{ac} \ll \nu \times m$, $\xi^s \perp \nu \times m$ and $\xi = \xi^{ac} + \xi^s$. Then we have a decomposition of $\eta = (\mu + \xi^{ac}) + \xi^s$ as a sum of one absolutely continuous measure and a singular one (both with respect to $\nu \times m$). On the other hand, notice that $\varphi_*(\nu \times m) \ll \nu \times m$ (it follows from the invariance of ν and by the non-singularity of $f(\theta, \cdot)$, for every $\theta \in \mathbb{X}$).

The previous Lemma 4.11 ensures that $\mu + \xi^{ac}$ is an absolutely continuous φ -invariant measure.

We claim that $\mu + \xi^{ac}$ is a non-zero finite measure. It suffices to prove that there exists $\gamma > 0$ such that, $\mu_n(\mathbb{X} \times I_0) > \gamma$ for all n big enough. Using that α^{-1} is invariant by ν and defining the family $s_j(\theta) := m(H_j(\theta, \varsigma) \cap Z(\theta, \lambda))$ of measurable functions for $j \geq 1$, we have for all $n \in \mathbb{Z}^+$,

$$\begin{aligned} \mu_n(\mathbb{X} \times I_0) &= \int_{\mathbb{X}} \frac{1}{n} \sum_{j=1}^n m(H_j(\alpha^{-j}(\theta), \varsigma) \cap Z(\alpha^{-j}(\theta), \lambda)) d\nu(\theta) = \frac{1}{n} \sum_{j=1}^n \int_{\mathbb{X}} s_j \circ \alpha^{-j}(\theta) d\nu(\theta) \\ &= \frac{1}{n} \sum_{j=1}^n \int_{\mathbb{X}} s_j(\theta) d\nu(\theta) = \int_{\mathbb{X}} \frac{1}{n} \sum_{j=1}^n m(H_j(\theta, \varsigma) \cap Z(\theta, \lambda)) d\nu(\theta). \end{aligned}$$

By Corollary 4.5 this last integral is bounded away from zero, as long as the set $Z(\lambda)$ has positive $\nu \times m$ -measure. More precisely, we have $\mu_n(Z(\lambda)) \geq \frac{\xi}{2}(\nu \times m)(Z(\lambda))$ for all big enough n . Hence $\mu + \xi^{ac}$ satisfies the conditions of the statement. □

4.3. Proof of the technical result. Here we present a proof of Theorem 4.1. The proof in the setting (\widetilde{H}_4^*) is similar to the proof on the setting (\widetilde{H}_4) . The result on the setting (\widetilde{H}_4) corresponds to Theorem B in [30], but here we do not assume the equicontinuity of the sequence $\{f_k\}$. For completeness, we prove the result on the setting (\widetilde{H}_4^*) and we remark the modifications for the proof on the setting (\widetilde{H}_4) .

4.3.1. Definitions and fundamental lemmas. In order to simplify the notation we say that $f^j(x) \in V_\epsilon \mathcal{D}$ if $f^j(x) \in V_\epsilon \mathcal{D}_j$ for $j \in \mathbb{N}$. By the recurrence property on the setting $(\widetilde{H_4}^*)$ (see equation (4.1)) we have that

Lemma 4.13. *Given $\gamma > 0$, there exists $\epsilon > 0$ such that for n big enough,*

$$(4.7) \quad \frac{1}{n} \sum_{j=0}^{n-1} \chi_{V_\epsilon \mathcal{D}}(f^j(x)) < \gamma$$

for any $x \in I_0$.

Remark 4.14. Note that the lemma also holds on setting $(\widetilde{H_4})$. In this case, (4.7) holds for any x such that $\log |Df^n(x)| > \lambda n$ and ϵ depends on λ . We use the equicontinuity of the sequence $\{f_k'\}$ instead of condition (4.1).

On the other hand, since the derivative of the maps of the sequence $\{f_k\}$ is bounded from above outside of the set of discontinuities, it holds the following result.

Lemma 4.15. *Given $\epsilon > 0$ and $l \in \mathbb{N}$, there exists $\delta > 0$ such that for any subinterval $J \subset I_0$,*

$$\text{if } |J| \leq 2\delta \text{ and } f_i^j(J) \cap \mathcal{D}_{i+j} = \emptyset \text{ for } 0 \leq j < k \quad \text{then} \quad |f_i^k(J)| < \epsilon$$

for all $i \geq 0$ and $0 < k \leq l$.

Proof. Let us consider $\Gamma = \sup\{|Df_k(x)|; k \in \mathbb{N}, x \notin \mathcal{D}_k\}$. The lemma follows from the next claim: for all $i, k \geq 0$, for any interval $J \subset I_0$, if $f_i^j(J) \cap \mathcal{D}_{i+j} = \emptyset$ for $0 \leq j < k$, then $|f_i^k(J)| \leq \Gamma^k |J|$. \square

Remark 4.16. Notice that this lemma also holds on the setting $(\widetilde{H_4})$, replacing the set \mathcal{D} by \mathcal{C} .

The main part in the proof of Theorem 4.1 is the control of the Lebesgue measure of the sets $Y_n(\lambda) \cap A_n(\{f_k\}, \delta)$, where $Y_n(\lambda) = \{x, \log |Df^n(x)| > \lambda n\}$ and

$$A_n(\delta) = A_n(\{f_k\}, \delta) := \left\{ x \in I_0; \frac{1}{n} \sum_{i=1}^n r_i(\{f_k\}, x) < \delta^2, \quad r_n(\{f_k\}, x) > 0 \right\},$$

for $n \in \mathbb{N}$ and $\delta > 0$ (and r_i as was defined in Section 4). For simplicity, we denote by $A_n(\delta)$ the set $A_n(\{f_k\}, \delta)$ and by $r_i(x)$ the number $r_i(\{f_k\}, x)$.

We introduce the following sets. For $\delta > 0$, $a_i \in \{0, 1\}$ for $i = 1, 2, \dots, n$,

$$C_\delta(a_1, a_2, \dots, a_n) := \{x \in I_0; r_i(x) \geq \delta \text{ if } a_i = 1, \quad 0 < r_i(x) < \delta \text{ if } a_i = 0\}$$

Note that for every $x \in A_n(\delta)$, there exist a_1, \dots, a_n (with $a_i \in \{0, 1\}$ for $i = 1, \dots, n$) and J component of $C_\delta(a_1, a_2, \dots, a_n)$ such that $x \in J$.

The key lemma in the proof of Theorem 4.1 is the following. Let $\#X$ denotes the number of components of X .

Lemma 4.17. *Given $\lambda > 0$, there exists $\delta > 0$ such that $\sum \#C_\delta(a_1, \dots, a_n) \leq \exp(n\lambda/2)$, where the sum is over all a_1, \dots, a_n such that $a_1 + a_2 + \dots + a_n < \delta n$. Moreover, the dependence of δ is as ς on the statement of Theorem 4.1.*

We need to decompose the interval I_0 set in a convenient way. Given $\epsilon > 0$, $m \leq n$, $\{t_1, \dots, t_m\} \subset \{0, 1, \dots, n-1\}$, we define

$$K_{n,\epsilon}(t_1, \dots, t_m) = \{x \in I_0; f^j(x) \in V_\epsilon \mathcal{D} \text{ if and only if } j \in \{t_1, \dots, t_m\}\}$$

By Lemma 4.13 we conclude that given $\gamma > 0$, there exists $\epsilon > 0$ such that for n big enough,

$$I_0 = \bigcup_{m=0}^n \bigcup_{t_1, \dots, t_m} K_{n,\epsilon}(t_1, \dots, t_m)$$

where the second union is over all subsets $\{t_1, \dots, t_m\} \subset \{0, 1, \dots, n-1\}$.

Let us denote by $\# \{I \subset C_\delta(a_1, \dots, a_n); I \cap K_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\}$ the number of components of $C_\delta(a_1, \dots, a_n)$ whose intersection with $K_{n,\epsilon}(t_1, \dots, t_m)$ is non empty.

From the last equation we conclude that

$$(4.8) \quad \sum_{a_1, \dots, a_n} \#C_\delta(a_1, \dots, a_n) \leq \sum_{a_1, \dots, a_n} \sum_{t_1, \dots, t_m} \#\{I \subset C_\delta(a_1, \dots, a_n); I \cap K_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\}$$

where the first sum is over all a_1, \dots, a_n such that $a_1 + \dots + a_n < \delta n$ and the second sum is over all subsets $\{t_1, \dots, t_m\} \subset \{0, 1, \dots, n-1\}$ with $m < \gamma n$.

Remark 4.18. In the setting (\widetilde{H}_4) , we count the number of components of $C_\delta(a_1, \dots, a_n)$ whose intersection with $Y_n(\lambda)$ is non empty. In order to do it, instead of the sets $K_{n,\epsilon}(t_1, \dots, t_m)$, we use the sets $Y_{n,\epsilon}(t_1, \dots, t_m) := Y_n(\lambda) \cap K_{n,\epsilon}(t_1, \dots, t_m)$.

4.3.2. *Components of $C_\delta(a_1, \dots, a_s)$.* We state some claims related to the number of components of the sets $C_\delta(a_1, \dots, a_n)$. Recall that p is the maximum number of elements in any \mathcal{D}_k (for $k \geq 0$). Given $I \subset I_0$ and $s \in \mathbb{N}$, we say $f^s(I) \cap \mathcal{D} = \emptyset$ (resp. $\neq \emptyset$) if $f^s(I) \cap \mathcal{D}_s = \emptyset$ (resp. $\neq \emptyset$).

Claim 4.19. For any a_1, a_2, \dots, a_s with $a_j \in \{0, 1\}$ for all j ,

$$\#C_\delta(a_1, \dots, a_s, 0) + \#C_\delta(a_1, \dots, a_s, 1) \leq 3(p+1)\#C_\delta(a_1, \dots, a_s)$$

Proof. Let I be a component of $C_\delta(a_1, \dots, a_s)$. If $f^s(I) \cap \mathcal{D} = \emptyset$ and $I' \subset I$ is a component of $C_\delta(a_1, \dots, a_s, 0)$, it can not exist one component of $C_\delta(a_1, \dots, a_s, 1)$ at each side of I' . So, there exist at most two components of $C_\delta(a_1, \dots, a_s, 0)$ in I . Hence, I is divided at most in 3 components of $C_\delta(a_1, \dots, a_s, 0) \cup C_\delta(a_1, \dots, a_s, 1)$.

If $f^s(I) \cap \mathcal{D} \neq \emptyset$, I is divided at most in $p+1$ components. Each one of these components have a boundary which goes by f^s to \mathcal{D} and is divided (as for the last case) at most in 3 components of $C_\delta(a_1, \dots, a_s, 0) \cup C_\delta(a_1, \dots, a_s, 1)$. \square

Claim 4.20. Let $s, n \in \mathbb{N}$ and J be a component of $C_\delta(a_1, \dots, a_s, 0)$. If $f^{s+i}(J) \cap \mathcal{D} = \emptyset$ for $1 \leq i \leq n$, then

$$\#\{I \subseteq C_\delta(a_1, \dots, a_s, 0^{i+1}), I \subseteq J\} \leq i+1.$$

for $1 \leq i \leq n$, where 0^{i+1} means that the last $i+1$ terms are equal to 0.

Proof. For $i=1$ the proof is contained on the proof of Claim 4.19. Now, note that every component of $C_\delta(a_1, \dots, a_s, 0^i)$ gives rise to one or two components of $C_\delta(a_1, \dots, a_s, 0^{i+1})$. The proof of Claim 4.20 follows by induction on i , showing that at most one component of $C_\delta(a_1, \dots, a_s, 0^i)$ gives rise to two components of $C_\delta(a_1, \dots, a_s, 0^{i+1})$. \square

To bound the number of components whose intersection with $K_{n,\epsilon}(t_1, \dots, t_m)$ is non-empty, we have the following claim.

Claim 4.21. Let $l \in \mathbb{N}$ and $\epsilon > 0$ be constants and let $\delta = \delta(l)$ be the number given by Lemma 4.15. For any a_1, \dots, a_s with $a_j \in \{0, 1\}$, $\{t_1, \dots, t_m\} \subset \{0, 1, \dots, n-1\}$. If $\{s+1, \dots, s+i\} \cap \{t_1, \dots, t_m\} = \emptyset$ and $i \leq l$, then

$$\#\{I \subseteq C_\delta(a_1, \dots, a_s, 0^{i+1}), I \cap K_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\} \leq (i+1)\#\{I \subseteq C_\delta(a_1, \dots, a_s, 0), I \cap K_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\}.$$

Proof. Let I be a component of $C_\delta(a_1, \dots, a_s, 0)$. Then $|f^{s+1}(I)| \leq 2\delta$. Let $i_0 \in \{1, 2, \dots, i\}$ the first number such that $f^{s+i_0}(I) \cap \mathcal{D} \neq \emptyset$. Since $f^{s+j}(I) \cap \mathcal{D} = \emptyset$ for $0 \leq j < i_0$, by Lemma 4.15, $|f^{s+i_0}(I)| < \epsilon$. Then, for all $x \in I$, $f^{s+i_0}(x) \in V_\epsilon \mathcal{D}$. Since $\{s+1, \dots, s+i\} \cap \{t_1, \dots, t_m\} = \emptyset$, then $I \cap K_{n,\epsilon}(t_1, \dots, t_m) = \emptyset$. Hence, if $I \cap K_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset$ and $\{s+1, \dots, s+i\} \cap \{t_1, \dots, t_m\} = \emptyset$, then $f^{s+j}(I) \cap \mathcal{D} = \emptyset$ for all $0 \leq j \leq i$, with $i \leq l$. Thus, claim follows using Claim 4.20. \square

4.3.3. *Proof of Lemma 4.17.* Given $m < n$, $\delta > 0$ and $\epsilon > 0$, let us consider a_1, \dots, a_n with $a_i \in \{0, 1\}$ (such that $a_1 + a_2 + \dots + a_n < \delta n$) and $\{t_1, \dots, t_m\} \subset \{0, \dots, n-1\}$. components of $C_\delta(a_1, \dots, a_n)$ whose intersection with $Y_{n,\epsilon}(t_1, \dots, t_m)$ is non-empty. We can decompose the sequence $a_1 \dots a_n$ in maximal blocks of 0's and 1's. We write the symbol ξ in the j -th position if $a_j = 1$ or $a_j = 0$ and $j = t_k$ for some $k \in \{1, \dots, m\}$. In this way we have,

$$(4.9) \quad a_1 a_2 \dots a_n = \xi^{i_1} 0^{j_1} \xi^{i_2} 0^{j_2} \dots \xi^{i_h} 0^{j_h}$$

with $0 \leq i_k, j_k \leq n$ for $k = 1, \dots, h$, $\sum_{k=1}^h (i_k + j_k) = n$ and $\sum_{k=1}^h i_k < m + \delta n$.

Lets us assume that a_1, \dots, a_n are as in (4.9). Let l, ϵ and δ be as in Lemma 4.15. Using claims 4.19 and 4.21 we have,

$$\begin{aligned} \#\{I \subset C_\delta(a_0, \dots, a_n), I \cap K_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\} &\leq \\ &\leq (3(p+1)(l+1)^{\frac{i_h}{\gamma}+1}(3(p+1))^{i_h} \dots (3(p+1)(l+1)^{\frac{i_1}{\gamma}+1}(3(p+1))^{i_1}) \\ &\leq (3(p+1))^{\sum_{k=1}^h i_k} (3(p+1))^h (l+1)^{\frac{\sum_{k=1}^h i_k}{\gamma}+h} \\ &\leq (3(p+1))^{m+\delta n+h} (l+1)^{\frac{m}{\gamma}+h}. \end{aligned}$$

Therefore, if $a_1 + a_2 + \dots + a_n < \delta n$ and $m < \gamma n$ we conclude from the inequality above that for n big enough,

$$\begin{aligned} \#\{I \subset C_\delta(a_1, \dots, a_n), I \cap K_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\} & \\ (4.10) \quad &\leq (3(p+1))^{\gamma n + \delta n} (3(p+1))^{2(\delta+\gamma)n} (l+1)^{\frac{m}{\gamma}+2(\delta+\gamma)n} \leq \exp(n \psi_0(l, \gamma, \delta)) \end{aligned}$$

where $\psi_0(l, \gamma, \delta) = 3(\delta + \gamma) \log(3(p+1)) + 2(\delta + \gamma + \frac{1}{\gamma}) \log(2l)$.

Using (4.10) and Stirling's formula in equation (4.8), we conclude that

$$\sum_{a_1, \dots, a_n} \#C_\delta(a_1, \dots, a_n) \leq \exp(n \psi_3(l, \gamma, \delta))$$

where $\psi_3(l, \gamma, \delta) = \psi_0(l, \gamma, \delta) + \psi_1(\gamma) + \psi_2(\delta)$, $\psi_1(\gamma) \rightarrow 0$ and $\psi_2(\delta) \rightarrow 0$ when $\gamma \rightarrow 0$ and $\delta \rightarrow 0$, respectively. Hence, we have to choose l such that

$$(4.11) \quad \frac{2}{l} \log(2l) < \frac{\lambda}{14}$$

and, let $\gamma > 0$ be such that

$$(4.12) \quad 2\gamma \log(2l) < \frac{\lambda}{14}, \quad 3\gamma \log(3(p+1)) < \frac{\lambda}{14}, \quad \text{and} \quad \psi_1(\gamma) < \frac{\lambda}{14}$$

Next, we find $\epsilon > 0$, using Lemma 4.13. Finally, given ϵ and l , let $\delta > 0$ be the constant given by Lemma 4.15 and satisfying

$$(4.13) \quad 2\delta \log(2l) < \frac{\lambda}{14}, \quad 3\delta \log(3(p+1)) < \frac{\lambda}{14} \quad \text{and} \quad \psi_3(\delta) < \frac{\lambda}{14}$$

With this choice, $\psi_3(l, \gamma, \delta) \leq \frac{\lambda}{2}$. Hence the first part of Lemma 4.17 is proved. On the other hand, observe that the choice of δ is given fundamentally by Lemmas 4.13 and 4.15. Namely, δ depends on: the constant λ in the definition of $Y_n(\lambda)$; the uniformity of ϵ (given $\ell \in \mathbb{N}$) on the equation (4.1); the uniform boundedness of $|Df_k|$ on the proof of Lemma 4.15; and the uniform boundedness of the number of discontinuity points for f_k , where $k \geq 0$. This concludes the proof of Lemma 4.17. \square

4.3.4. *Proof of Theorem 4.1.* Note that for every $N \in \mathbb{N}$ it holds

$$E \cap \left(\bigcap_{n \geq N} Y_n(\lambda) \right) \cap \mathbb{C} \left(\bigcup_{n \geq N} A_n(\delta) \cap Y_n(\lambda) \right) \subset E \cap \left(\bigcap_{n \geq N} \mathbb{C} A_n(\delta) \cap Y_n(\lambda) \right).$$

where $\mathbb{C}B$ denotes the complement set of B and $|B|$ denotes the Lebesgue measure of B . By the hypotheses of theorem, $|E \cap (\cap_{n \geq N} Y_n(\lambda))|$ converges to the Lebesgue measure of E . On the other hand, note that if J is a component of $C_\delta(a_1, \dots, a_n)$ (with $a_1 + \dots + a_n < \delta n$) then $|J \cap Y_n(\lambda) \cap A_n(\delta)| \leq |I_0| \exp(-n\lambda)$. Then, using Lemma 4.17 we conclude that $|\cup_{n \geq N} A_n(\delta) \cap Y_n(\lambda)|$ converges to zero when $N \rightarrow \infty$. Therefore, $|\cap_{n \geq N} (\mathbb{C} A_n(\delta) \cap Y_n(\lambda)) \cap E|$ converges to $|E|$ when $N \rightarrow \infty$. Thus, we conclude that (4.2) holds considering $3\zeta = \delta^2$. \square

5. NON-INVERTIBLE BASE TRANSFORMATION

Let $\varphi : \mathbb{X} \times I_0 \rightarrow \mathbb{X} \times I_0$ or $\varphi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \times \mathbb{Y}$ be a skew-product satisfying (H_2) and the remaining conditions of Theorems A or D. We define a natural extension $\hat{\varphi}$ of this map and we prove that it satisfies (H_1) , (H_2^*) and also the remaining conditions of the statement of the Main Theorems.

5.1. Inverse limit construction. We use a standard construction which allows to define, for an endomorphism of a measure space, an induced invertible bimeasurable map of a new measure space. For more details, see for instance [12, Chapter 10.4]. We perform the construction with the map $\alpha : \mathbb{X} \rightarrow \mathbb{X}$.

First consider the (inverse limit) space $\hat{\mathbb{X}}$ which is formed by points

$$\hat{\theta} = (\theta_0, \theta_{-1}, \theta_{-2}, \dots),$$

where $\theta_{-i} \in \mathbb{X}$ for $i \geq 0$ and $\alpha(\theta_{-i}) = \theta_{-i+1}$ for $i \geq 1$. Then we have

- (1) $\mathbb{X}^{\mathbb{N}}$ with the product topology is a metrizable space (see [17, Lemma 111.15]);
- (2) $\mathbb{X}^{\mathbb{N}}$ is separable (see [17, Theorems 111.14 and 58.7]);
- (3) as a topological space (in fact, a metrizable space), $\mathbb{X}^{\mathbb{N}}$ admits the Borel σ -algebra $\mathcal{B}_{\mathbb{X}^{\mathbb{N}}}$, which is the σ -algebra generated by the open sets of the product topology on $\mathbb{X}^{\mathbb{N}}$;
- (4) the product σ -algebra $\prod_{i \in \mathbb{N}} \mathcal{B}_{\mathbb{X}_i}$ on $\mathbb{X}^{\mathbb{N}}$ coincides with $\mathcal{B}_{\mathbb{X}^{\mathbb{N}}}$. ($\mathbb{X}_i = \mathbb{X}$ for all $i \in \mathbb{N}$);
- (5) as subset of $\mathbb{X}^{\mathbb{N}}$, $\hat{\mathbb{X}}$ is endowed with the product topology, and therefore has a Borel σ -algebra $\mathcal{B}_{\hat{\mathbb{X}}}$;
- (6) $\mathcal{B}_{\hat{\mathbb{X}}}$ coincides with the σ -algebra obtained by intersecting $\prod_{i \in \mathbb{N}} \mathcal{B}_{\mathbb{X}_i}$ with $\hat{\mathbb{X}}$.

Now, $\hat{\mathbb{X}}$ with the σ -algebra $(\prod_{i \in \mathbb{N}} \mathcal{B}_{\mathbb{X}_i}) \cap \hat{\mathbb{X}}$ is a measurable space. For the sets of the form

$$(A)_n = \{\hat{\theta} = (\theta_0, \theta_{-1}, \theta_{-2}, \dots) \in \hat{\mathbb{X}}; \theta_{-n} \in A\}$$

where $A \in \mathcal{B}_{\mathbb{X}}$ and $n \geq 0$, we define $\hat{\nu}((A)_n) = \nu(A)$. Since these sets generate the σ -algebra and the conditions of compatibility of Kolmogorov's Theorem are satisfied, we have a measure $\hat{\nu}$ defined on the σ -algebra.

We can consider the map $\hat{\alpha} : \hat{\mathbb{X}} \rightarrow \hat{\mathbb{X}}$ given by

$$\hat{\alpha}((\theta_0, \theta_{-1}, \theta_{-2}, \dots)) = (\alpha(\theta_0), \alpha(\theta_{-1}), \alpha(\theta_{-2}), \dots) = (\alpha(\theta_0), \theta_0, \theta_{-1}, \theta_{-2}, \dots).$$

This map is invertible $\hat{\alpha}^{-1}((\theta_0, \theta_{-1}, \theta_{-2}, \dots)) = (\theta_{-1}, \theta_{-2}, \theta_{-3}, \dots)$. The measure $\hat{\nu}$ is invariant with respect to $\hat{\alpha}$.

Therefore we have constructed an invertible map $\hat{\alpha}$, bimeasurable (with the Borel σ -algebra $\mathcal{B}_{\hat{\mathbb{X}}}$) on a metric space $\hat{\mathbb{X}}$, such that $\pi_0 \circ \hat{\alpha}(\hat{\theta}) = \alpha \circ \pi_0(\hat{\theta})$ for every $\hat{\theta} \in \hat{\mathbb{X}}$, where $\pi_0(\hat{\theta}) = \theta_0$. It is also useful to define the natural projection map $P : \hat{\mathbb{X}} \times I_0 \rightarrow \mathbb{X} \times I_0$, by $P(\hat{\theta}, x) = (\pi_0(\hat{\theta}), x) = (\theta_0, x)$.

5.2. Non-invertible base. Let us define the map $\hat{\varphi} : \hat{\mathbb{X}} \times I_0 \rightarrow \hat{\mathbb{X}} \times I_0$, $\hat{\varphi}(\hat{\theta}, x) = (\hat{\alpha}(\hat{\theta}), \hat{f}(\hat{\theta}, x))$, where $\hat{f}(\hat{\theta}, x) = f(\theta_0, x)$. Since $\hat{\alpha}$, P and f are measurable then $\hat{\varphi}$ is measurable, i.e. $\hat{\varphi}^{-1}(\mathcal{B}_{\hat{\mathbb{X}}} \times \mathcal{B}_{I_0}) \subset \mathcal{B}_{\hat{\mathbb{X}}} \times \mathcal{B}_{I_0}$.

Note that the set of critical and discontinuity points for $\hat{f}_{\hat{\theta}}$ projects onto the corresponding set for f_{θ_0} . Hence the measurability of the set

$$\hat{\mathcal{S}} = \{(\hat{\theta}, x) \in \hat{\mathbb{X}} \times I_0 : x \in \mathcal{C}_{\theta_0} \cup \mathcal{D}_{\theta_0}\} = P^{-1}(\mathcal{S})$$

follows from the measurability of the set \mathcal{S} and of the map P . Thus, $\hat{\varphi}$ satisfies condition (H_1) .

We note that the set of discontinuity points $\mathcal{D}_{\hat{\alpha}}$ of $\hat{\alpha}$ coincides with the set

$$(\mathcal{D}_{\alpha})_0 = \{\hat{\theta} \in \hat{\mathbb{X}}; \theta_0 \in \mathcal{D}_{\alpha}\}$$

and so $\hat{\nu}(\mathcal{D}_{\hat{\alpha}}) = \nu(\mathcal{D}_{\alpha}) = 0$. On the other hand, for the map $\hat{F} : \hat{\mathbb{X}} \rightarrow B(I_0)$, $\hat{\theta} \mapsto \hat{f}_{\hat{\theta}} = f_{\theta_0}$ we have that $\mathcal{D}_{\hat{F}} \subset (\mathcal{D}_F)_0$. Hence the map $\hat{\varphi}$ satisfies conditions (H_2^*) and (H_3) . The map $\hat{\varphi}$ clearly satisfies condition (H_4) (resp. (H_4^*)) if the map φ satisfies the condition (H_4) (resp. (H_4^*)).

Moreover, if φ has positive Lyapunov exponents along the vertical direction according to $\nu \times m$, on the subset Z , then $\hat{\varphi}$ has positive Lyapunov exponents along the vertical direction according to $\hat{\nu} \times m$, on the subset $P^{-1}Z$. It also holds that $\hat{\nu} \times m(P^{-1}Z) = \nu \times m(Z)$.

Thus, $\hat{\varphi}$ is a skew-product in the conditions of Theorems A and D, and Corollary B.

We remark that in order to prove the relative compactness of the sequences of measures $\{\eta_n\}$ and $\{\mu_n\}$ (see Lemma 4.6 and Remark 4.7) we use the fact that \mathbb{X} is a separable metrizable and complete topological space. The space $\hat{\mathbb{X}}$ can fail to be complete. To solve this problem, we can consider $\hat{\nu}$ as a measure defined on $\mathbb{X}^{\mathbb{N}}$ (stating that $\hat{\nu}(\mathbb{X}^{\mathbb{N}} \setminus \hat{\mathbb{X}}) = 0$). Thus, we can find a closed set $\mathbb{X}_1 \subset \hat{\mathbb{X}}$ such

that $\hat{\nu}(\hat{\mathbb{X}} \setminus \mathbb{X}_1) < \epsilon$. On the other hand, since $\mathbb{X}^{\mathbb{N}}$ is a separable metrizable and complete topological space, we can find a compact set $\mathbb{X}_2 \subset \mathbb{X}^{\mathbb{N}}$, such that $\hat{\nu}(\mathbb{X}^{\mathbb{N}} \setminus \mathbb{X}_2) < \epsilon$. Hence, for the compact set $\mathbb{X}_1 \cap \mathbb{X}_2$, we have that $\hat{\nu}(\hat{\mathbb{X}} \setminus (\mathbb{X}_1 \cap \mathbb{X}_2)) < 2\epsilon$. Therefore, considering $\mathbb{X}_0 = \mathbb{X}_1 \cap \mathbb{X}_2$ in Lemma 4.6 and Remark 4.7, the relative compactness of the sequences of measures $\{\eta_n\}$ and $\{\mu_n\}$ follows.

Hence we may repeat the same sequence of steps in the arguments in Section 4 assuming Theorem 4.1 to conclude the result in Proposition 4.12: there exists an invariant measure $\hat{\mu}$ which is absolutely continuous with respect to $\hat{\nu} \times \mathfrak{m}$, with $\hat{\mu}(P^{-1}(Z(\lambda))) > 0$. Now we push this measure for the original space $\mathbb{X} \times I_0$.

Lemma 5.1. *$P_*\hat{\mu}$ is an φ -invariant measure which is absolutely continuous with respect to $\nu \times \mathfrak{m}$, and $P_*\hat{\mu}(Z(\lambda)) > 0$.*

Proof. Let $A \subset \mathbb{X} \times I_0$ be a measurable subset. Using that $\varphi \circ P = P \circ \hat{\varphi}$ and the $\hat{\varphi}$ -invariance of $\hat{\mu}$ we have that

$$P_*\hat{\mu}(\varphi^{-1}(A)) = \hat{\mu}(P^{-1}\varphi^{-1}(A)) = \hat{\mu}((P \circ \hat{\varphi})^{-1}(A)) = \hat{\mu}(\hat{\varphi}^{-1}(P^{-1}A)) = P_*\hat{\mu}(A)$$

and then $P_*\hat{\mu}$ is invariant with respect to φ .

On the other hand, if $(\nu \times \mathfrak{m})(A) = 0$, then $(\hat{\nu} \times \mathfrak{m})(P^{-1}A) = 0$. Using the absolute continuity of $\hat{\mu}$, we conclude that $P_*\hat{\mu}(A) = 0$. □

6. FINITELY MANY ERGODIC BASINS

Here we conclude the proofs of Theorem A and Corollary B, proving that the invariant sets with positive $\nu \times \mathfrak{m}$ -measure, have mass bounded away from zero.

Given $\lambda > 0$, let $Z(\lambda) \subset \mathbb{X} \times I_0$ the set of points with vertical Lyapunov exponents greater than 2λ , i.e., points for which the limit in equation (1.2) is greater than 2λ .

Proposition 6.1. *Given $\lambda > 0$, there exists $b > 0$ such that every φ -invariant subset $G \subset Z(\lambda)$ with positive $\nu \times \mathfrak{m}$ -measure satisfies $(\nu \times \mathfrak{m})(G) > b$.*

This ensures that the ergodic basins $B_i = B(\mu_i)$ of the measures provided by Corollary B has $\nu \times \mathfrak{m}$ -measure uniformly bounded away from zero. Since these are pairwise disjoint subsets, their number in a finite measure space must be finite.

For the proof of Proposition, we need the following result.

Lemma 6.2. *Given $\varsigma > 0$, there exists $K > 0$ such that, for any $i \in \mathbb{N}$ and any $(\theta, x) \in \mathbb{X} \times I_0$, if $r_i(\theta, x) > \varsigma$, there exists $J_i(x)$ such that $f_{\theta}^i(J_i(x)) = B(f_{\theta}^i(x), \varsigma/2)$, f_{θ}^i restricted to $J_i(x)$ is a C^3 diffeomorphism and*

$$(6.1) \quad \frac{1}{K} \leq \frac{|Df_{\theta}^i(y)|}{|Df_{\theta}^i(z)|} \leq K \quad \text{for all } y, z \in J_i(x).$$

Proof. Let $(\theta, x) \in \mathbb{X} \times I_0$ such that $r_i(\theta, x) > \varsigma$. By definition of r_i , there exists $T_i \subset I_0$ such that $x \in T_i$, f_{θ}^i restricted to T_i is a C^3 diffeomorphism and the connected components of $f_{\theta}^i(T_i) \setminus \{f_{\theta}^i(x)\}$ have length $> \varsigma$. Let us choose $J_i(x) \subset T_i$ such that $f_{\theta}^i(J_i(x)) = B(f_{\theta}^i(x), \varsigma/2)$. Note that $f_{\theta}^i(T_i)$ contains an $1/2$ -scaled neighborhood of $f_{\theta}^i(J_i(x))$. It means that both connected components of $f_{\theta}^i(T_i) \setminus f_{\theta}^i(J_i(x))$ have length at least $|f_{\theta}^i(J_i(x))|/2$. By Koebe Principle (see [23, Theorem IV.1.2]), there exists K such that (6.1) holds. The distortion $K > 0$ does not depend on the point (θ, x) , nor the iterate i . □

Proof of Proposition 6.1. Let $G \subset Z(\lambda)$ be a forward φ -invariant set, such that $(\nu \times \mathfrak{m})(G) > 0$. Given $\lambda > 0$, let us consider the constant $\varsigma > 0$ given by Theorem 4.1. Let $K > 0$ the constant found on Lemma 6.2. Denoting by $G(\theta)$ the θ -section of G , i.e, $G(\theta) := \{x \in I_0; (\theta, x) \in G\}$, let us define the measurable set

$$B_G^{\varsigma} := \left\{ \theta \in \mathbb{X}, \mathfrak{m}(G(\theta)) \geq \frac{\varsigma}{4K} \right\}$$

Since the measure ν is ergodic for the map α , then

$$(6.2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{B_G^c}(\alpha^i(\theta)) = \int \chi_{B_G^c} d\nu$$

for all θ in a ν -full measure set. Let $\theta_0 \in \mathbb{X}$ be a point such that $m(G(\theta_0)) > 0$ and (6.2) holds for $\theta = \theta_0$. By Theorem 4.1 applied to the set $E = G(\theta_0)$, we can find a point $x_0 \in G(\theta_0)$ such that $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n r_i(\theta_0, x_0) \geq 3\varsigma$. We can assume that x_0 is a density point of $G(\theta_0)$. Thus, there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$,

$$\frac{m(G(\theta_0) \cap B(x_0, \epsilon))}{m(B(x_0, \epsilon))} \geq \frac{1}{2}$$

On the other hand, we can find $N \in \mathbb{N}$ such that $\sum_{i=1}^n r_i(\theta_0, x_0) \geq 2\varsigma n$ and $\log |Df_{\theta_0}^n(x_0)| \geq \lambda n$ for all $n \geq N$. Then, if $r_i(\theta_0, x_0) \geq \varsigma$, for $i \geq N$, the interval $J_i(x_0)$ found on Lemma 6.2 is such that $|J_i(x_0)| \leq K\varsigma e^{-\lambda i}$. Therefore, there exists $n_0 \geq N$ such that $J_i(x_0) \subset B(x_0, \epsilon_0)$, provided $i \geq n_0$ (and obviously only when $r_i(\theta_0, x_0) \geq \varsigma$).

Claim 6.3. *If $r_i(\theta_0, x_0) \geq \varsigma$ and $i \geq n_0$ then $\alpha^i(\theta_0) \in B_G^c$.*

Proof. Let $J_i^*(x_0)$ the maximal ball centered at x_0 contained in $J_i(x_0)$. Using Lemma 6.2,

$$\frac{m(f_{\theta_0}^i(J_i^*(x_0) \cap G(\theta_0)))}{m(f_{\theta_0}^i(J_i^*(x_0)))} \geq \frac{1}{K} \frac{m(J_i^*(x_0) \cap G(\theta_0))}{m(J_i^*(x_0))} \geq \frac{1}{2K}$$

for $i \geq n_0$. Then we have that $m(f_{\theta_0}^i(J_i^*(x_0) \cap G(\theta_0))) \geq \varsigma/4K$. Since $f_{\theta_0}^i(G(\theta_0)) \subset G(\alpha^i(\theta_0))$ (by the forward φ -invariance of G), the claim follows. \square

An immediate consequence of the claim is that for all $n \geq n_0$,

$$\sum_{i=n_0}^n \chi_{B_G^c}(\alpha^i(\theta_0)) \geq \# \{n_0 \leq i \leq n; r_i(\theta_0, x_0) \geq \varsigma\}$$

Now, using Pliss Lemma (see Lemma 4.4), there exists $\xi = \xi(\varsigma) > 0$ such that for $n \geq n_0$,

$$\frac{\# \{1 \leq i \leq n; r_i(\theta_0, x_0) \geq \varsigma\}}{n} \geq \xi$$

since $\sum_{i=1}^n r_i(\theta_0, x_0) \geq 2\varsigma n$, for $n \geq n_0$. Hence the limit in (6.2) for $\theta = \theta_0$ is greater than ξ . It means that $\nu(B_G^c) \geq \xi$. Thus, Proposition 6.1 follows considering $b = \varsigma\xi/4K$. \square

Finally we can conclude the proof of Theorem A and Corollary B.

Proof of Corollary B. By assumption, there exists $\lambda > 0$ such that $Z(\lambda)$ has full $(\nu \times m)$ -measure. Let μ_0 be the φ -invariant probability measure absolutely continuous with respect to $\nu \times m$ given by Proposition 4.12 and Lemma 5.1. Considering the normalized restriction to the forward invariant set $Z(\lambda)$, we can assume that $\mu_0(Z(\lambda)) = 1$. Since every invariant set, with positive $\nu \times m$ -measure, has $\nu \times m$ -measure greater than b (by Proposition 6.1), we can decompose μ_0 in a finite number of ergodic components. Then $\mu_0 = \sum_{i=1}^s a_i \mu_i$, where $a_i > 0$, $\sum_{i=1}^s a_i = 1$ and μ_i are ergodic φ -invariant absolutely continuous probability measures.

If $Z_1 = Z(\lambda) \setminus \cup_{i=1}^s B(\mu_i)$ still has positive $\nu \times m$ -measure, then we can repeat the arguments of Section 4 for the set $Z_1 \subset Z(\lambda)$ instead of $Z(\lambda)$. Repeating this argument, we obtain the ergodic components as in the statement of Corollary B such that $\nu \times m$ -a.e. point in $Z(\lambda)$ is in the basin of one of these measure. The number of such measures is finite, since the basin of each of them is a collection of pairwise disjoint invariant sets with $\nu \times m$ -positive measure, and Proposition 6.1 holds. \square

Proof of Theorem A. Since $Z = \cup_{n \in \mathbb{N}} Z(1/n)$, the previous argument applied to each $Z(1/n)$ provides finitely many ergodic probability measures whose basins cover $Z(1/n)$, for each $n \geq 1$. This concludes the proof of Theorem A. \square

7. SRB MEASURES FOR RANDOM NON-UNIFORMLY EXPANDING MAPS

Let $(\mathbb{X}, \mathcal{B}_\mathbb{X}, \nu, \alpha, f)$ be an admissible random non-uniformly expanding map on I_0 . Let us consider the associated skew-product φ defined on $\mathbb{X} \times I_0$. By Corollary B, there exist μ_1, \dots, μ_t , φ -invariant ergodic probabilities, such that $(\nu \times m)$ -a.e. (θ, x) is in the basin of one of these measures. Denote by B_i the ergodic basin $B(\mu_i)$ of the measure μ_i , for $1 \leq i \leq t$. As usual, $B_i(\theta)$ denotes the θ -section of the set B_i .

Proof of Theorem C. Define p_i as the projection on I_0 of μ_i . By a straightforward calculation, we can prove that $RB_\theta(p_i) \supseteq B_i(\theta)$. As μ_i is absolutely continuous with respect to $\nu \times m$, then $\nu \times m(B_i) > 0$. Since B_i is φ -invariant and ν is α -ergodic, then $m(B_i(\theta)) > 0$ for ν -almost every $\theta \in \mathbb{X}$. It implies that p_i is a SRB probability for the random dynamical system.

Since $m(I_0 \setminus \cup_{i=1}^t B_i(\theta)) = 0$, for ν -almost every $\theta \in \mathbb{X}$, then ν -almost surely, the union of the random basins of p_1, \dots, p_t has total Lebesgue measure. Clearly, these measures are absolutely continuous with respect to Lebesgue measure. \square

8. HIGHER DIMENSIONAL FIBERS

Here we outline the arguments in the higher-dimensional fiber case. The strategy is the same as the one presented for one-dimensional fibers. We start by considering the sequences $\eta_n(\theta)$ and η_n as in Section 3.

Then we use the notion of hyperbolic times from [5] to redefine the sets $\mathcal{H}(\sigma)$ and redefine $\mu_n(\theta)$ replacing $H_n(\theta, \varsigma)$ by $H_n(\sigma)$. Finally we just have to obtain the analogous results to Corollary 4.5 and Lemmas 4.8 and 4.9.

The argument then follows the proof of Theorem A through Lemmas 4.10 and 4.11, whose proof uses the statements of the results in Section 3.

In what follows, since hyperbolic times have been extensively investigated recently, we cite most of the results from other published works.

Given $0 < \sigma < 1$ and $b, \delta > 0$, we say that the positive integer n is a (σ, δ, b) -hyperbolic time for $(\theta, x) \in \mathbb{X} \times \mathbb{Y}$ if

$$(8.1) \quad \prod_{j=n-k}^{n-1} \|Df_{\alpha^j(\theta)}(f_\theta^j(x))^{-1}\| \leq \sigma^k \quad \text{and} \quad \text{dist}_\delta(f_\theta^k(x), \mathcal{S} \cap (\{\alpha^k(\theta)\} \times \mathbb{Y})) \geq e^{-bk} \quad \text{for } k = 0, \dots, n-1.$$

We now outline the properties of these special times. For detailed proofs see [5, Proposition 2.8] and [4, Proposition 2.6, Corollary 2.7, Proposition 5.2].

Proposition 8.1. *There are constants $C_1, \delta_1 > 0$ depending on (σ, δ, b) and φ only such that, if n is (σ, δ, b) -hyperbolic time for (θ, x) , then there are hyperbolic preballs $V_k(\theta, x)$ which are neighborhoods of $f_\theta^{n-k}(x)$ on $\{\alpha^{n-k}(\theta)\} \times \mathbb{Y}$, $k = 1, \dots, n$, such that*

- (1) $f_{\alpha^{n-k}(\theta)}^k \mid V_k(\theta, x)$ maps $V_k(\theta, x)$ diffeomorphically to the ball of radius δ_1 around $f_\theta^n(x)$ inside $\{\alpha^n(\theta)\} \times \mathbb{Y}$;
- (2) for every $1 \leq k \leq n$ and $y, z \in V_k(\theta, x)$

$$\text{dist}(f_{\alpha^k(\theta)}^{n-k}(y), f_{\alpha^k(\theta)}^{n-k}(z)) \leq \sigma^{k/2} \cdot \text{dist}(f_\theta^n(y), f_\theta^n(z));$$

- (3) for $y, z \in V_k(\theta, x)$

$$\frac{1}{C_1} \leq \frac{|\det Df_{\alpha^k(\theta)}^{n-k}(y)|}{|\det Df_{\alpha^k(\theta)}^{n-k}(z)|} \leq C_1.$$

The following ensures existence of infinitely many hyperbolic times for Lebesgue almost every point for non-uniformly expanding maps with slow recurrence to the singular set. A complete proof can be found in [5, Section 5].

Theorem 8.2. *Let $\varphi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \times \mathbb{Y}$ be as in the statement of Theorem D, i.e., non-uniformly expanding along the fibers.*

Then there are $\sigma \in (0, 1)$, $\delta, b > 0$ and there exists $\rho = \rho(\sigma, \delta, b) > 0$ such that $\nu \times \text{Leb}$ -a.e. $(\theta, x) \in \mathbb{X} \times \mathbb{Y}$ has infinitely many (σ, δ, b) -hyperbolic times. Moreover if we write $0 < n_1 < n_2 < n_3 < \dots$ for the hyperbolic times of (θ, x) , then their asymptotic frequency satisfies

$$\liminf_{N \rightarrow \infty} \frac{\#\{k \geq 1 : n_k \leq N\}}{N} \geq \rho \quad \text{for } \nu \times \text{Leb} \text{-a.e. } (\theta, x) \in \mathbb{X} \times \mathbb{Y}.$$

Now we define, in this setting

$$\mathcal{H}_n(\sigma, \delta, b) := \{(\theta, x) \in \mathbb{X} \times \mathbb{Y} : n \text{ is a } (\sigma, \delta, b)\text{-hyperbolic time for } (\theta, x)\}$$

and, having fixed σ, δ, b according to Theorem 8.2, we set

$$\mathcal{H}_n(\theta) := (\{\theta\} \times \mathbb{Y}) \cap \mathcal{H}_n(\sigma, \delta, b).$$

Now we are able to state and prove the analogous result to Corollary 4.5 with the same arguments.

Lemma 8.3. *Let $E \subset \mathbb{X} \times \mathbb{Y}$ be such that $(\theta, x) \in E$ satisfies*

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\varphi^j(\theta)}(f_{\theta}^j(x))^{-1}\| < -\omega < 0.$$

If n is big enough we have $\int \frac{1}{n} \sum_{i=1}^n \text{Leb}(\mathcal{H}_i(\theta)) \, d\nu(\theta) \geq \frac{\rho}{2}(\nu \times \text{Leb})(E)$.

Here we assume the measurability of $H_n(\theta)$ in what follows. This will be proved in Appendix A.

We define the measures $\mu_n(\theta)$ on \mathbb{Y} , for every $\theta \in \mathbb{X}$ and every $n \in \mathbb{N}$, precisely as in (4.3) and, using them, for every $n \in \mathbb{N}$ we define the measures μ_n on $\mathbb{X} \times \mathbb{Y}$ as in (4.4). We need to show that these measures are well-defined. This is proved in Appendix A.

Lemma 8.4. *There exists $K > 0$ such that for any measurable subset $A \subset \mathbb{Y}$,*

$$\mu_n(\theta)(A) \leq K \cdot \text{Leb}(A)$$

for every $\theta \in \mathbb{X}, n \in \mathbb{N}$. Moreover, K depends only on the distortion constant provided by Proposition 8.1.

The proof of Lemma 8.4 is essentially the same proof of Lemma 4.8 replacing I_0 by \mathbb{Y} and the respective σ -algebras throughout. The reader can check the details following [4, Proposition 5.2].

The analogous statements to Lemmas 4.9 and 4.10 are proved in the same way. At this point, assuming Lemma 8.4, we have the analogous results to Corollary 4.5 and Lemmas 4.8 and 4.9. The rest of the argument proving the existence of absolutely continuous invariant measures is entirely analogous. We also obtain a similar statement to Proposition 4.12.

For the ergodic decomposition, the arguments are the same as in Section 6.

8.1. Non-invertible base map with higher-dimensional fibers. With the notation introduced in Section 5, we define the map $\hat{\varphi} : \hat{\mathbb{X}} \times \mathbb{Y} \rightarrow \hat{\mathbb{X}} \times \mathbb{Y}$, $\hat{\varphi}(\hat{\theta}, x) = (\hat{\alpha}(\hat{\theta}), \hat{f}(\hat{\theta}, x))$, where $\hat{f}(\hat{\theta}, x) = f(\theta_0, x)$. In the exact same manner as in Section 5, we deduce that this map satisfies conditions $(H_1), (H_2^*)$ and (H_3) , if φ satisfies conditions (H_1) through (H_3) .

Moreover the argument about relative compactness and the proofs of Lemmas 4.6 and 5.1 need no change. We are left to show that if φ is non-uniformly expanding along the fibers, then $\hat{\varphi}$ is likewise. But this follows from

- the easy observation that $\hat{\varphi}^k(\hat{\theta}, x) = (\sigma^k(\hat{\theta}), f_{\theta_0}^k(x))$;
- together with the fact that the full $\nu \times \text{Leb}$ -measure subset W of $\mathbb{X} \times \mathbb{Y}$ satisfying the conditions (1.3) and (1.4) of non-uniform expansion and slow recurrence provides the set $\hat{W} = \pi^{-1}(W)$ which also has full $\hat{\nu} \times \text{Leb}$ -measure on $\hat{\mathbb{X}} \times \mathbb{Y}$.

So the points $(\hat{\theta}, x) \in \hat{W}$ will satisfy conditions (1.3) and (1.4). Hence $\hat{\varphi}$ is non-uniformly expanding along the fibers, with a bijection $\hat{\alpha}$ as the base transformation.

We can now apply the same arguments of Sections 3 and 4 to $\hat{\varphi}$. So our main results also hold if we replace condition (H_2^*) by condition (H_2) .

APPENDIX A. MEASURABILITY

Here we prove that the measures η_n defined on Section 3 together with the measures μ_n defined on Section 4 are well-defined. We consider separately the case with one dimensional fibers and the case with higher dimensional fibers.

A.1. The measures η_n are well defined. By the Hahn Extension Theorem, it is enough to define the measures on rectangles $A \times J$ with $A \in \mathcal{B}_{\mathbb{X}}$ and $J \in \mathcal{B}_{I_0}$. It easily follows from

Proposition A.1. *Let $J \subset I_0$ be a Borel set. For every $n \in \mathbb{N}$, the function $\mathbb{X} \ni \theta \mapsto \eta_n(\theta)(J)$ is measurable.*

Proof. Let us fix a set $J \in \mathcal{B}_{I_0}$. To prove the measurability of $\theta \mapsto \eta_n(\theta)(J)$ it suffices to prove the measurability of the functions $\theta \mapsto \eta_J^i(\theta) := (f_{\alpha^{-i}(\theta)}^i)_* m(J)$, for $i \in \mathbb{N}$. Let us define the following functions

$$\begin{aligned} \alpha^{-1} \times id : \mathbb{X} \times I_0 &\rightarrow \mathbb{X} \times I_0 & \pi_{\mathbb{X}} : \mathbb{X} \times I_0 &\rightarrow \mathbb{X} & \pi_{I_0} : \mathbb{X} \times I_0 &\rightarrow I_0 \\ (\theta, x) &\mapsto (\alpha^{-1}(\theta), x) & (\theta, x) &\mapsto \theta & (\theta, x) &\mapsto x \end{aligned}$$

and χ_J is the characteristic function of J . The projection maps are clearly measurable, considering on $\mathbb{X} \times I_0$ the σ -algebra $\mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}$. Since compositions of measurable maps are measurable maps, $\alpha^{-1} \times id(\theta, x) = (\alpha^{-1} \circ \pi_{\mathbb{X}}(\theta, x), \pi_{I_0}(\theta, x))$ is also measurable.

With these notations, we have that $\eta_J^i(\theta) = \int_{I_0} \phi_i(\theta, x) d m(x)$, where $\phi_i : \mathbb{X} \times I_0 \rightarrow \mathbb{R}$ is defined by

$$(A.1) \quad (\theta, x) \mapsto \phi_i(\theta, x) := \chi_J \circ \pi_{I_0} \circ \phi^i \circ (\alpha^{-i} \times id)^i(\theta, x).$$

Using Fubini's Theorem, the measurability of ϕ_i (considering the σ -algebra $\mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}$) implies the measurability of $\theta \mapsto \eta_J^i(\theta)$ (considering the σ -algebra $\mathcal{B}_{\mathbb{X}}$). \square

A.2. The measures μ_n are well defined. We assume the skew-product satisfies the property (H_4) . The proof for the case of (H_4^*) is entirely analogous. It is enough to substitute \mathcal{C} by \mathcal{D} .

As in the case of η_n , the well-definition of the measures μ_n follows from Hahn Extension Theorem and the following result which implies that these measures are defined on the algebra of the rectangles.

Proposition A.2. *Let $J \subset I_0$ be a borelian set. For every $n \in \mathbb{N}$, the function $\mathbb{X} \ni \theta \mapsto \mu_n(\theta)(J)$ is measurable.*

In the definition of the measures μ_n appear the sets $H_j(\theta, \zeta)$ ($j \in \mathbb{N}, \theta \in \mathbb{X}$). These sets depend on the maps $r_j(\theta, x)$ and $l_j^*(\theta, x) := |f_{\theta}^j(T^j(\theta, x))|$. We study first the measurability of these functions.

Let us recall the definition of the function r_i (given in Section 4). Given $i \in \mathbb{N}$ and a point $(\theta, x) \in \mathbb{X} \times I_0$, we denote by $T_i(\theta, x)$ the maximal interval such that $f_{\theta}^j(T_i(\theta, x)) \cap \mathcal{C}_{\alpha^j(\theta)} = \emptyset$ for all $j < i$. Thus $r_i(\theta, x)$ denotes the minimum of the lengths of the connected components of $f_{\theta}^i(T_i(\theta, x) \setminus \{x\})$.

Lemma A.3. *The maps $r_i : \mathbb{X} \times I_0 \rightarrow \mathbb{R}$ are measurable, for all $i \in \mathbb{N}$.*

Proof. For fixed $\theta \in \mathbb{X}$, $x \mapsto r_i(\theta, x)$ is a continuous function, since f_{θ}^i (for $\theta \in \mathbb{X}, i \in \mathbb{N}$) are piecewise continuous C^3 maps. Hence, by [21, Lemma 9.2], we conclude r_i is measurable, if for fixed $x \in I_0$ the function $\theta \mapsto r_i(\theta, x)$ is measurable. We claim that this last condition is true. To prove it, we write $r_i(\cdot, x)$ as a composition of measurable maps.

For $i \in \mathbb{N}$, let us define the set

$$\mathcal{C}^i = \bigcup_{j=0}^{i-1} \varphi^{-j} \mathcal{C} \cup (\mathbb{X} \times \partial I_0)$$

Given $(\theta, x) \in \mathbb{X} \times I_0$, the interval $T_i(\theta, x) = (a_i(\theta, x), b_i(\theta, x))$ can be defined in the following way

$$\begin{aligned} a_i(\theta, x) &= \sup(E^{x-})_{\theta} := \sup\{y \in I_0; (\theta, y) \in E^{x-}\} \\ b_i(\theta, x) &= \inf(E^{x+})_{\theta} := \inf\{y \in I_0; (\theta, y) \in E^{x+}\} \end{aligned}$$

where $E^{x-} = (\mathbb{X} \times (-\infty, x] \cap I_0) \cap \mathcal{C}^i$ and $E^{x+} = (\mathbb{X} \times [x, +\infty) \cap I_0) \cap \mathcal{C}^i$. The sets E^{x-} and E^{x+} are measurable, since by hypotheses (H_1) , \mathcal{C} is measurable. Then, for fixed $x \in I_0$, the measurability of the functions $\theta \mapsto a_i(\theta, x)$ and $\theta \mapsto b_i(\theta, x)$ follows from the next result.

Claim A.4. *Let E be a set in $\mathcal{B}_{\mathbb{X}} \times \mathcal{B}_{I_0}$ and let $S : \mathbb{X} \rightarrow I_0, s : \mathbb{X} \rightarrow I_0$ be functions defined by $S(\theta) = \sup E_\theta = \sup\{y \in I_0; (\theta, y) \in E\}, s(\theta) = \inf E_\theta = \inf\{y \in I_0; (\theta, y) \in E\}$. Then S and s are measurable maps.*

Proof. We prove first for the map S . Let $b \in \mathbb{R}$ be a constant. We want to prove that $S^{-1}((b, +\infty)) \in \mathcal{B}_{\mathbb{X}}$. First, let us suppose that E is an open set on $\mathbb{X} \times I_0$. Let θ_0 be any point in $S^{-1}((b, +\infty))$. Then there exists $y_0 \in I_0$ such that $y_0 > b$ and $(\theta_0, y_0) \in E$. The openness of E shows the existence of open sets $A \subset \mathbb{X}$ and $B \subset I_0$ such that $(\theta_0, y_0) \in A \times B \subset E$. Thus $A \subset S^{-1}((b, +\infty))$ and it shows that $S^{-1}((b, +\infty))$ is an open set.

In the general case, given any measurable set E , let us consider the sets

$$B\left(E, \frac{1}{n}\right) = \left\{z \in \mathbb{X} \times I_0; \text{dist}(z, w) < \frac{1}{n} \text{ for some } w \in E\right\}.$$

for $n \in \mathbb{N}$. We consider the functions $S_n(\theta) = \sup\{y \in I_0; (\theta, y) \in B(E, 1/n)\}$. These functions are measurable by what we have proved. Since $S = \inf_{n \in \mathbb{N}} S_n$, the measurability of S follows. \square

Using the measurability of $a_i(\theta, x)$ and $b_i(\theta, x)$ we conclude the measurability of $\theta \mapsto r_i(\theta, x)$ (all for fixed $x \in I_0$). It finishes the proof of Lemma A.3. \square

Now, we want to prove the measurability of the maps l_j^* . Let us consider a sequence of measurable partitions $\dots \subset \mathcal{P}_{n+1} \subset \mathcal{P}_n \subset \dots \subset \mathcal{P}_1$ of I_0 such that the norm of \mathcal{P}_n is less than $1/n$. Choose a point x_i^n in each P_i^n element of \mathcal{P}_n and define the functions

$$l_j^n(\theta, x) := |f_\theta^j(T^j(\theta, x_i^n))| \text{ for all } x \in P_i^n.$$

We also consider the map $l_j := \liminf_{n \rightarrow \infty} l_j^n$.

Lemma A.5. *The maps $l_j : \mathbb{X} \times I_0 \rightarrow \mathbb{R}$ are measurable for all $j \in \mathbb{N}$.*

Proof. For fixed $x \in I_0$, the maps $\theta \mapsto |f_\theta^j(T^j(\theta, x))|$ are measurable, since

$$|f_\theta^j(T^j(\theta, x))| = |\pi_{I_0} \circ \varphi^i \circ (id, a_i(\cdot, x))(\theta) - \pi_{I_0} \circ \varphi^i \circ (id, b_i(\cdot, x))(\theta)|.$$

Therefore the maps l_j^n are measurable. Obviously it implies the measurability of maps l_j . \square

Proof of Proposition A.2. By Lemma A.5, the map l_j is measurable and $l_j(\theta, x) = l_j^*(\theta, x)$ if $r_j(\theta, x) > 0$. By Lemma A.3, the sets $\mathcal{H}_i(\sigma) := \{z \in \mathbb{X} \times I_0; r_i(z) > \sigma\}$ are measurable, for any $\sigma > 0$. These facts imply that $H_i(\sigma) = \mathcal{H}_i(\sigma) \cap (l_j^*)^{-1}(3\sigma, \infty)$ is a measurable set.

Let us fix a set $J \in \mathcal{B}_{I_0}$. As on Proposition A.1, to prove the measurability of $\theta \mapsto \mu_n(\theta)(J)$ it suffices to prove the measurability of the functions $\theta \mapsto \mu_j^i(\theta) := (f_{\alpha^{-i}(\theta)}^i)_*(m|H_i(\alpha^{-i}(\theta), \zeta) \cap Z(\alpha^{-i}(\theta), \lambda))(J)$, for $i \in \mathbb{N}$. Now, we have that $\mu_j^i(\theta) = \int_{I_0} \phi_i(\theta, x) \psi_i(\theta, x) d m(x)$, where $\phi_i, \psi_i : \mathbb{X} \times I_0 \rightarrow \mathbb{R}$, ϕ_i are respectively defined in (A.1) and

$$(\theta, x) \mapsto \psi_i(\theta, x) := \chi_{H_i(\zeta)} \circ (\alpha^{-1} \times id)^i(\theta, x) \cdot \chi_{Z(\lambda)} \circ (\alpha^{-1} \times id)^i(\theta, x)$$

Once again, using Fubini's Theorem, the measurability of $(\theta, x) \mapsto \phi_i(\theta, x) \psi_i(\theta, x)$ implies the measurability of $\theta \mapsto \mu_j^i(\theta)$. \square

A.3. Higher-dimensional fibers.

A.3.1. *The measures η_n are well defined.* This case is precisely the same as the case with one-dimensional fibers, so we have nothing to add.

A.3.2. *The measures μ_n are well defined.* From the definition of μ_n in the higher dimensional case, we see that it is enough to show that for every $n \in \mathbb{N}$ and Borel set $S \subset \mathbb{Y}$ the function $\mathbb{X} \ni \theta \mapsto \mu_n(\theta)(S)$ is measurable. For this it is enough to prove the following.

Lemma A.6. *The function $\mathbb{X} \ni \theta \mapsto \text{Leb}(\mathcal{H}_j(\alpha^{-j}(\theta)) \cap (f_{\alpha^j(\theta)}^j)^{-1}(S))$ is measurable for each fixed $j \in \mathbb{N}$ and measurable $S \subset \mathbb{Y}$.*

Analogously to the previous subsection, we consider the maps

$$\begin{aligned} \alpha^{-1} \times id : \mathbb{X} \times \mathbb{Y} &\rightarrow \mathbb{X} \times \mathbb{Y} & \pi_{\mathbb{X}} : \mathbb{X} \times \mathbb{Y} &\rightarrow \mathbb{X} & \pi_{I_0} : \mathbb{X} \times \mathbb{Y} &\rightarrow \mathbb{Y} \\ (\theta, x) &\mapsto (\alpha^{-1}(\theta), x) & (\theta, x) &\mapsto \theta & (\theta, x) &\mapsto x \end{aligned}$$

and χ_S the characteristic function of S . These functions are all measurable with respect to the corresponding Borel σ -algebras. We consider also $\chi_{\mathcal{H}_n}$ the characteristic function of $\mathcal{H}_n(\sigma, \delta, b)$.

Lemma A.7. *The set $\mathcal{H}_n(\sigma, \delta, b)$ is a Borel subset of $\mathbb{X} \times \mathbb{Y}$.*

Proof. According to the definition of (σ, δ, b) -hyperbolic time

$$\mathcal{H}_n(\sigma, \delta, b) = \{(\theta, x) \in \mathbb{X} \times \mathbb{Y} : (8.1) \text{ is true for } (\theta, x)\}$$

is an intersection of at most finitely many sets of the form $\{(\theta, x) \in \mathbb{X} \times \mathbb{Y} : g(\theta, x) > c\}$ for a measurable function $g : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ and some constant $c \in \mathbb{R}$. Indeed, if we define for $k = 0, \dots, n-1$

$$g_k(\theta, x) := \prod_{j=n-k}^{n-1} \|Df_{\alpha^j(\theta)}(f_{\theta}^j(x))^{-1}\| \quad \text{and} \quad d_k(\theta, x) := \text{dist}_{\delta}(f_{\theta}^k(x), \mathcal{S} \cap (\{\alpha^k(\theta)\} \times \mathbb{Y})),$$

then we can write

$$\mathcal{H}_n(\sigma, \delta, b) = \{(\theta, x) \in \mathbb{X} \times \mathbb{Y} : g_k(\theta, x) < \sigma^k \quad \text{and} \quad d_k(\theta, x) > e^{-bk}, k = 0, \dots, n-1\}.$$

Thus $\mathcal{H}_n(\sigma, \delta, b)$ is a Borel subset of $\mathbb{X} \times \mathbb{Y}$ as soon as we show that g_k, d_k are measurable functions for each $k \geq 0$.

Clearly g_k is measurable from condition (H_6) . For the functions $d_k : \mathbb{X} \times \mathbb{Y} \rightarrow [0, +\infty)$ we clearly have

$$d_k(\theta, x) = D(\alpha^k(\theta), f_{\theta}^k(x)) \quad \text{where} \quad D(\theta, x) = \inf \xi_{(\theta, x)}$$

and we define

$$\xi(\theta, x, y) = \xi_{(\theta, x)}(y) := \text{dist}_{\delta}(x, y) \cdot \chi_{\mathcal{S}}(\theta, y) + \delta \cdot (1 - \chi_{\mathcal{S}}(\theta, y)).$$

Clearly $\xi : \mathbb{X} \times \mathbb{Y} \times \mathbb{Y} \rightarrow [0, \delta]$ is measurable, so $D : \mathbb{X} \times \mathbb{Y} \rightarrow [0, \delta]$ is also measurable and d_k is a composition of D with other measurable maps from condition (H_5) . This completes the argument showing that $\mathcal{H}_n(\sigma, \delta, b)$ is a Borel subset of $\mathbb{X} \times \mathbb{Y}$. \square

Now we are ready to prove the first lemma.

Proof of Lemma A.6. We note that we can write

$$(A.2) \quad \text{Leb}(\mathcal{H}_j(\alpha^{-j}(\theta)) \cap (f_{\alpha^j(\theta)}^j)^{-1}(S)) = \int \phi_j(\theta, x) \psi_j(\theta, x) d \text{Leb}(x),$$

where

$$\phi_j(\theta, x) := \chi_S \circ \pi_{I_0} \circ \varphi^j \circ (\alpha^{-j} \times id)^j(\theta, x) \quad \text{and} \quad \psi_j(\theta, x) := \chi_{\mathcal{H}_n} \circ (\alpha^{-j} \times id)^j(\theta, x).$$

Since both ϕ_j and ψ_j are Borel measurable from $\mathbb{X} \times \mathbb{Y}$ to \mathbb{R} , Fubini's Theorem ensures that (A.2) is a measurable function of $\theta \in \mathbb{X}$, as we need. This concludes the proof. \square

With Lemma A.6 we complete the proof of the measurability of all functions used in the previous sections.

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V. ARAUJO, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DA BAHIA, AV. ADEMAR DE BARROS S/N, 40170-110 SALVADOR, BRAZIL.

E-mail address: vitor.d.araujo@ufba.br

JAVIER SOLANO, INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL FLUMINENSE, RUA MÁRIO SANTOS BRAGA, S/N, VALONGUINHO, 24.020-140 NITERÓI, RJ-BRAZIL

E-mail address: jsolano@impa.br